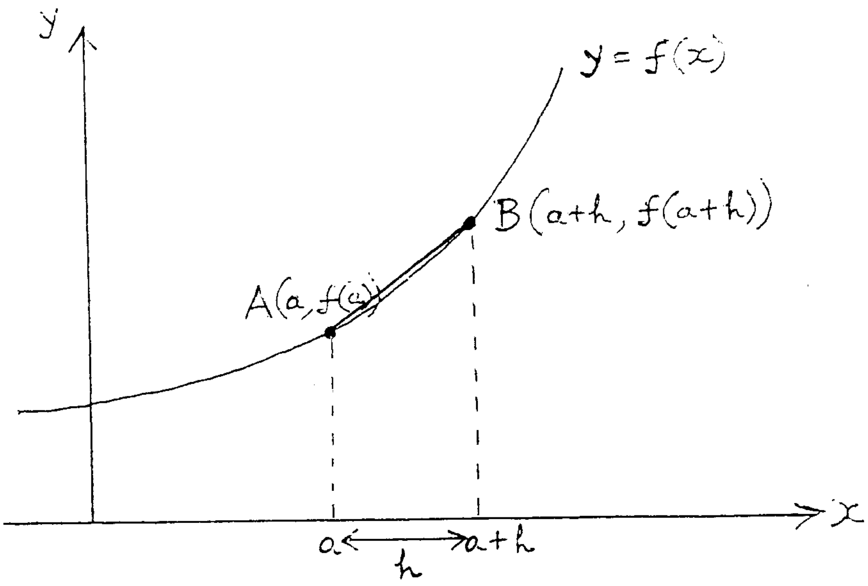
BASIC DIFFERENTIATION

DIFFERENTIATION FROM FIRST PRINCIPLES

Recall that $f'(a)$ is the gradient of the tangent to the curve $y = f(x)$ at the point where $x = a$.



$$m_{AB} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

If h is small, the gradient of the chord AB will be approximately equal to the gradient of the tangent at A. As h gets smaller, the approximation becomes more accurate.

As $h \to 0$, $\dfrac{f(a+h) - f(a)}{h} \to f'(a)$.

We write $f'(a) = \lim_{h \to 0} \left\{ \dfrac{f(a+h) - f(a)}{h} \right\}$.

Replacing a with x gives a formula for finding $f'(x)$:

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

Using the above formula to find $f'(x)$ is known as **differentiation from first principles**.

Worked Example 1

Find the derivative of the function $f(x) = 3x^2$ **from first principles.**

Solution

$$f(x) = 3x^2$$

$$\begin{aligned} f(x+h) = 3(x+h)^2 &= 3(x^2 + 2xh + h^2) \\ &= 3x^2 + 6xh + 3h^2 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} \\ &= \frac{6xh + 3h^2}{h} \\ &= 6x + 3h \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\} \\ &= \lim_{h \to 0} \{6x + 3h\} = 6x \end{aligned}$$

Worked Example 2

Find the derivative of the function $f(x) = 2x^2 - 3x + 1$ **from first principles**.

Solution

$$f(x) = 2x^2 - 3x + 1$$

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 3(x+h) + 1 \\ &= 2(x^2 + 2xh + h^2) - 3(x+h) + 1 \\ &= 2x^2 + 4xh + 2h^2 - 3x - 3h + 1 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 1 - (2x^2 - 3x + 1)}{h} \\ &= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 1 - 2x^2 + 3x - 1}{h} \\ &= \frac{4xh + 2h^2 - 3h}{h} \\ &= 4x + 2h - 3 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\} \\ &= \lim_{h \to 0} \{4x + 2h - 3\} = 4x - 3 \end{aligned}$$

THE PRODUCT RULE

If $y = uv$, where u and v are function of x, then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This rule is known as the **product rule** and is used to differentiate products of functions. When using the product rule, answers should always be simplified as far as possible.

Worked Example 1

Given $y = x^3 \sin x$, find $\frac{dy}{dx}$.

Solution

$y = x^3 \sin x$

This must be differentiated using the **product rule**.

$u = x^3$ $\quad\quad$ $v = \sin x$

$\Rightarrow \quad \frac{du}{dx} = 3x^2$ $\quad\quad$ $\frac{dv}{dx} = \cos x$

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x^3 \cos x + 3x^2 \sin x \\ &= x^2 (x \cos x + 3 \sin x)\end{aligned}$$

Worked Example 2

Given $y = x^4 \cos 2x$, find $\frac{dy}{dx}$.

Solution

$y = x^4 \cos 2x$

This must be differentiated using the **product rule**.

$$u = x^4 \qquad v = \cos 2x$$

$$\Rightarrow \quad \frac{du}{dx} = 4x^3 \qquad \frac{dv}{dx} = -2\sin 2x$$

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= -2x^4 \sin 2x + 4x^3 \cos 2x \\ &= 2x^3(2\cos 2x - x\sin 2x)\end{aligned}$$

Worked Example 3

Given $y = x\sin(x^2)$, find $\frac{dy}{dx}$.

Solution

$y = x\sin(x^2)$

This must be differentiated using the **product rule**.

$$u = x \qquad v = \sin(x^2)$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x\cos(x^2)$$

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= 2x^2\cos(x^2) + \sin(x^2)\end{aligned}$$

Worked Example 4

Given $y = x^2 \cos^4 x$, find $\frac{dy}{dx}$.

Solution

$y = x^2 \cos^4 x$

This must be differentiated using the **product rule**.

$u = x^2$

$\Rightarrow \quad \frac{du}{dx} = 2x$

$v = \cos^4 x = (\cos x)^4$

$\frac{dv}{dx} = 4(\cos x)^3 \cdot (-\sin x) = -4\cos^3 x \sin x$

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= -4x^2 \cos^3 x \sin x + 2x\cos^4 x \\ &= 2x\cos^3 x(\cos x - 2x\sin x)\end{aligned}$$

Worked Example 5

Find the derivative of the function $f(x) = x^2\sqrt{x^2+1}$.
Give your answer as a single fraction in its simplest form.

Solution

$f(x) = x^2\sqrt{x^2+1}$

This must be differentiated using the **product rule**.

$$u = x^2 \qquad\qquad v = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{du}{dx} = 2x \qquad\qquad \frac{dv}{dx} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

$$\begin{aligned} f'(x) &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x^2 \cdot \frac{x}{\sqrt{x^2+1}} + 2x\sqrt{x^2+1} \\ &= \frac{x^3}{\sqrt{x^2+1}} + \frac{2x\sqrt{x^2+1}}{1} \\ &= \frac{x^3}{\sqrt{x^2+1}} + \frac{2x\sqrt{x^2+1}\times\sqrt{x^2+1}}{1\times\sqrt{x^2+1}} \quad \text{[this step can be omitted with practice]} \\ &= \frac{x^3}{\sqrt{x^2+1}} + \frac{2x(x^2+1)}{\sqrt{x^2+1}} \\ &= \frac{x^3+2x^3+2x}{\sqrt{x^2+1}} \\ &= \frac{3x^3+2x}{\sqrt{x^2+1}} \\ &= \frac{x(3x^2+2)}{\sqrt{x^2+1}} \end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "DIFFERENTIATION: THE PRODUCT RULE".

THE QUOTIENT RULE

If $y = \frac{u}{v}$, where u and v are function of x (with $v \neq 0$), then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

This rule is known as the **quotient rule** and is used to differentiate quotients of functions. When using the quotient rule, answers should always be simplified as far as possible.

Worked Example 1

Given $y = \frac{2x}{x^2+1}$, find $\frac{dy}{dx}$.

Solution

This must be differentiated using the **quotient rule**.

$u = 2x \qquad\qquad v = x^2 + 1$

$\Rightarrow \quad \frac{du}{dx} = 2 \qquad\qquad \frac{dv}{dx} = 2x$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2} \\
&= \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} \\
&= \frac{2 - 2x^2}{(x^2+1)^2} \\
&= \frac{2(1-x^2)}{(x^2+1)^2} \\
&= \frac{2(1-x)(1+x)}{(x^2+1)^2}
\end{aligned}$$

Worked Example 2

Given $y = \frac{\sin x}{x^3}$, $x \neq 0$, find $\frac{dy}{dx}$.

Solution

This must be differentiated using the **quotient rule**.

$$u = \sin x \qquad\qquad v = x^3$$

$$\Rightarrow \quad \frac{du}{dx} = \cos x \qquad\qquad \frac{dv}{dx} = 3x^2$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\
&= \frac{x^3 \cos x - 3x^2 \sin x}{(x^3)^2} \\
&= \frac{x^2 (x\cos x - 3\sin x)}{x^6} \\
&= \frac{x\cos x - 3\sin x}{x^4}
\end{aligned}$$

Worked Example 3

Given $y = \dfrac{\sin x}{1+\cos x}$, $-\pi < x < \pi$, find $\dfrac{dy}{dx}$.

Solution

This must be differentiated using the **quotient rule**.

$$u = \sin x \qquad\qquad v = 1 + \cos x$$

$$\Rightarrow \quad \frac{du}{dx} = \cos x \qquad\qquad \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{\cos x(1+\cos x) - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2} \quad [\text{since } \cos^2 x + \sin^2 x = 1]$$

$$= \frac{1}{1+\cos x}$$

Worked Example 4

Given $y = \dfrac{x}{\sqrt{x+4}}$, $x > -4$, find $\dfrac{dy}{dx}$.

Solution

This must be differentiated using the **quotient rule**.

$$u = x \qquad\qquad v = \sqrt{x+4} = (x+4)^{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad\qquad \frac{dv}{dx} = \frac{1}{2}(x+4)^{-\frac{1}{2}} \cdot 1 = \frac{1}{2\sqrt{x+4}}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{\sqrt{x+4} - x \cdot \dfrac{1}{2\sqrt{x+4}}}{(\sqrt{x+4})^2}$$

$$= \frac{\sqrt{x+4} - \dfrac{x}{2\sqrt{x+4}}}{x+4}$$

$$= \frac{\sqrt{x+4} \times 2\sqrt{x+4} - \dfrac{x}{2\sqrt{x+4}} \times 2\sqrt{x+4}}{(x+4) \times 2\sqrt{x+4}}$$ [this step can be omitted with practice]

$$= \frac{2(x+4) - x}{2(x+4)^{\frac{3}{2}}}$$

$$= \frac{x+8}{2(x+4)^{\frac{3}{2}}}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "DIFFERENTIATION: THE QUOTIENT RULE".

DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

There are other trigonometric functions defined in terms of $\sin x$ and $\cos x$.

$$\tan x = \frac{\sin x}{\cos x}$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

Examples: (1) $\operatorname{cosec}\frac{\pi}{3} = \frac{1}{\sin\frac{\pi}{3}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

(2) $\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$

(3) $\cot^2\frac{\pi}{6} = \frac{1}{\tan^2\frac{\pi}{6}} = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{1}{\frac{1}{3}} = 3$

The function $\tan x$ can be differentiated using the quotient rule as follows:

Let $y = \tan x = \dfrac{\sin x}{\cos x}$.

$$u = \sin x \qquad\qquad v = \cos x$$

$$\Rightarrow \quad \frac{du}{dx} = \cos x \qquad\qquad \frac{dv}{dx} = -\sin x$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \quad [\text{since } \cos^2 x + \sin^2 x = 1] \\
&= \sec^2 x
\end{aligned}$$

The derivatives of the other trigonometric functions can be found in a similar way.

SUMMARY OF DERIVATIVES

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$

The above standard derivatives must be memorised. It is important to note that these derivatives are only valid when the angle x is measured in **radians**.

Example 1

$y = \tan 3x$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = \sec^2 3x \cdot \frac{d}{dx}(3x)$$
$$= 3\sec^2 3x$$

[With practice, the answer can be written down directly.]

Example 2

$y = \text{cosec}\, 4x$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = -\text{cosec}\, 4x \cot 4x \cdot \frac{d}{dx}(4x)$$
$$= -4\,\text{cosec}\, 4x \cot 4x$$

Example 3

$y = \sec^2 x = (\sec x)^2$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = 2\sec x \cdot \frac{d}{dx}(\sec x)$$
$$= 2\sec x \cdot \sec x \tan x$$
$$= 2\sec^2 x \tan x$$

Example 4

$y = \cot^3 2x = (\cot 2x)^3$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = 3(\cot 2x)^2 \cdot \frac{d}{dx}(\cot 2x)$$
$$= 3\cot^2 2x \cdot (-\text{cosec}^2 2x) \cdot 2$$
$$= -6\cot^2 2x\, \text{cosec}^2 2x$$

[Note that the chain rule was also used to differentiate $\cot 2x$.]

Example 5

$y = x^4 \sec 2x$

This must be differentiated using the **product rule**.

$$u = x^4 \qquad\qquad v = \sec 2x$$

$$\Rightarrow \quad \frac{du}{dx} = 4x^3 \qquad\qquad \frac{dv}{dx} = 2\sec 2x \tan 2x$$

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= 2x^4 \sec 2x \tan 2x + 4x^3 \sec 2x \\ &= 2x^3 \sec 2x(x \tan 2x + 2)\end{aligned}$$

Example 6

$$y = \frac{\tan x}{x^2}$$

This must be differentiated using the **quotient rule**.

$$u = \tan x \qquad\qquad v = x^2$$

$$\Rightarrow \quad \frac{du}{dx} = \sec^2 x \qquad\qquad \frac{dv}{dx} = 2x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{x^2 \sec^2 x - 2x \tan x}{(x^2)^2} \\ &= \frac{x(x \sec^2 x - 2\tan x)}{x^4} \\ &= \frac{x \sec^2 x - 2\tan x}{x^3}\end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS".

DIFFERENTIATION OF EXPONENTIAL FUNCTIONS

We find that:

$$\frac{d}{dx}(e^x) = e^x$$

This can be proved using differentiation from first principles as follows.

Let $f(x) = e^x$.

Then $f(x+h) = e^{x+h}$.

$$\frac{f(x+h) - f(x)}{h} = \frac{e^{x+h} - e^x}{h}$$

$$= \frac{e^x e^h - e^x}{h}$$

$$= \frac{e^x(e^h - 1)}{h}$$

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{e^x(e^h - 1)}{h} \right\}$$

$$= e^x \times \lim_{h \to 0} \left\{ \frac{e^h - 1}{h} \right\}$$

Numerical investigation shows that $\lim_{h \to 0} \left\{ \frac{e^h - 1}{h} \right\} = 1$.

Hence $f'(x) = e^x \times 1 = e^x$.

Example 1

$y = e^{6x}$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = e^{6x} \cdot \frac{d}{dx}(6x)$$
$$= 6e^{6x}$$

[With practice, the answer can be written down directly.]

Example 2

$y = 3e^{x^2} + 4e^{-2x}$

Each term must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = 3 \cdot e^{x^2} \cdot 2x + 4 \cdot e^{-2x} \cdot (-2)$$
$$= 6xe^{x^2} - 8e^{-2x}$$

Example 3

$y = e^{2x} \cos 2x$

This must be differentiated using the **product rule**.

$$u = e^{2x} \qquad v = \cos 2x$$
$$\Rightarrow \quad \frac{du}{dx} = 2e^{2x} \qquad \frac{dv}{dx} = -2\sin 2x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= -2e^{2x} \sin 2x + 2e^{2x} \cos 2x$$
$$= 2e^{2x}(\cos 2x - \sin 2x)$$

Example 4

$$y = \frac{x^2}{e^{2x}}$$

This must be differentiated using the **quotient rule**.

$$u = x^2 \qquad\qquad v = e^{2x}$$

$$\Rightarrow \quad \frac{du}{dx} = 2x \qquad\qquad \frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{2xe^{2x} - 2x^2e^{2x}}{(e^{2x})^2}$$

$$= \frac{2xe^{2x}(1-x)}{e^{4x}}$$

$$= \frac{2x(1-x)}{e^{2x}} \quad \text{[note that } \frac{e^{2x}}{e^{4x}} = \frac{1}{e^{2x}}\text{]}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "DIFFERENTIATION OF EXPONENTIAL FUNCTIONS".

DIFFERENTIATION OF LOGARITHMIC FUNCTIONS

Recall the following facts about natural logarithms:

(1) $\ln x = \log_e x$ where $e = 2 \cdot 71828...$

(2) $\ln x$ and e^x are **inverse functions**.

This means that:

(3) $e^{\ln x} = x \quad (x > 0)$

(4) $\ln(e^x) = x$

The derivative of $\ln x$ $(x > 0)$ can be found as follows.

Let $y = \ln x$ $(x > 0)$.

Then $y = \ln x \Rightarrow e^y = x$

$\Rightarrow x = e^y$

Differentiate both sides with respect to y $\Rightarrow \dfrac{dx}{dy} = e^y$

Now $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{e^y} = \dfrac{1}{x}$.

Hence:

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0)$$

It is important to realise that this result can only be used to differentiate $\ln x$, i.e. $\log_e x$. This result does not apply for differentiating logarithms to other bases (e.g. $\log_{10} x$). You will see later in the course how to differentiate logarithms to other bases.

Example 1

$y = \ln(3x - 1)$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = \frac{1}{3x-1} \cdot \frac{d}{dx}(3x - 1)$$
$$= \frac{1}{3x-1} \cdot 3$$
$$= \frac{3}{3x-1}$$

[With practice, the answer can be written down directly.]

Example 2

$y = \ln(4x^2 + 1)$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = \frac{1}{4x^2+1} \cdot 8x$$
$$= \frac{8x}{4x^2+1}$$

Example 3

$y = \ln(\cos 4x)$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = \frac{1}{\cos 4x} \cdot (-4\sin 4x)$$
$$= \frac{-4\sin 4x}{\cos 4x}$$
$$= -4\tan 4x$$

Example 4

$f(x) = \ln(e^{2x} + 1)$

This must be differentiated using the **chain rule**.

$$f'(x) = \frac{1}{e^{2x} + 1} \cdot 2e^{2x}$$
$$= \frac{2e^{2x}}{e^{2x} + 1}$$

Example 5

$y = x^3 \ln x$

This must be differentiated using the **product rule**.

$$u = x^3 \qquad v = \ln x$$
$$\Rightarrow \quad \frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x^3 \cdot \frac{1}{x} + 3x^2 \ln x$$
$$= x^2 + 3x^2 \ln x$$
$$= x^2(1 + 3\ln x)$$

Example 6

$$y = \frac{\ln x}{e^{2x}}$$

This must be differentiated using the **quotient rule**.

$$u = \ln x \qquad\qquad v = e^{2x}$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \qquad\qquad \frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{\frac{1}{x} \cdot e^{2x} - 2e^{2x}\ln x}{(e^{2x})^2}$$

$$= \frac{\frac{e^{2x}}{x} - 2e^{2x}\ln x}{e^{4x}}$$

$$= \frac{\frac{e^{2x}}{x} \times x - 2e^{2x}\ln x \times x}{e^{4x} \times x}$$ [this step can be missed out with practice]

$$= \frac{e^{2x} - 2xe^{2x}\ln x}{xe^{4x}}$$

$$= \frac{e^{2x}(1 - 2x\ln x)}{xe^{4x}}$$

$$= \frac{1 - 2x\ln x}{xe^{2x}}$$ [note that $\frac{e^{2x}}{e^{4x}} = \frac{1}{e^{2x}}$]

The laws of logarithms can sometimes be used to simplify functions before differentiation.

Recall the three laws of logarithms below:

(1)	$\ln(ab) = \ln a + \ln b$
(2)	$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
(3)	$\ln(a^n) = n\ln a$

Worked Example 1

Given $y = \ln(x^4 e^{2x})$, find $\frac{dy}{dx}$.

Solution

$$\begin{aligned} y &= \ln(x^4 e^{2x}) \\ &= \ln(x^4) + \ln(e^{2x}) \\ &= 4\ln x + 2x \qquad [\text{note that } \ln(e^{2x}) = 2x] \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 4 \cdot \frac{1}{x} + 2 \\ &= \frac{4}{x} + 2 \end{aligned}$$

Worked Example 2

Given $y = \ln\left(\frac{x}{2x+1}\right)$, $x > 0$, show that $\frac{dy}{dx} = \frac{1}{x(2x+1)}$.

Solution

$$
\begin{aligned}
y &= \ln\left(\frac{x}{2x+1}\right) \\
&= \ln x - \ln(2x+1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{x} - \frac{1}{2x+1}\cdot 2 \\
&= \frac{1}{x} - \frac{2}{2x+1} \\
&= \frac{1(2x+1)}{x(2x+1)} - \frac{2x}{x(2x+1)} \\
&= \frac{2x+1-2x}{x(2x+1)} \\
&= \frac{1}{x(2x+1)}
\end{aligned}
$$

[*Alternative Method*:

The function $y = \ln\left(\frac{x}{2x+1}\right)$ can be differentiated directly using the **chain rule**.

$$\frac{dy}{dx} = \frac{1}{\frac{x}{2x+1}}\cdot\frac{d}{dx}\left(\frac{x}{2x+1}\right) = \frac{2x+1}{x}\cdot\frac{d}{dx}\left(\frac{x}{2x+1}\right) \qquad \ldots(*)$$

The function $\frac{x}{2x+1}$ can now be differentiated using the **quotient rule**.

$$\frac{d}{dx}\left(\frac{x}{2x+1}\right) = \frac{(2x+1)\cdot 1 - x\cdot 2}{(2x+1)^2} = \frac{1}{(2x+1)^2}$$

From (*): $\quad \frac{dy}{dx} = \frac{2x+1}{x}\cdot\frac{1}{(2x+1)^2} = \frac{1}{x(2x+1)}$]

Worked Example 3

Given $y = \ln\sqrt{2x^3+1}$, $x > 0$, find $\frac{dy}{dx}$.

Solution

$$
\begin{aligned}
y &= \ln\sqrt{2x^3+1} \\
&= \ln(2x^3+1)^{\frac{1}{2}} \\
&= \frac{1}{2}\ln(2x^3+1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2}\cdot\frac{1}{2x^3+1}\cdot 6x^2 \\
&= \frac{3x^2}{2x^3+1}
\end{aligned}
$$

[*Alternative Method*:

The function $y = \ln\sqrt{2x^3+1}$ can be differentiated using the **chain rule**.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sqrt{2x^3+1}}\cdot\frac{d}{dx}\sqrt{2x^3+1} \\
&= \frac{1}{\sqrt{2x^3+1}}\cdot\frac{d}{dx}(2x^3+1)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2x^3+1}}\cdot\frac{1}{2}(2x^3+1)^{-\frac{1}{2}}\cdot 6x^2 \\
&= \frac{1}{\sqrt{2x^3+1}}\cdot\frac{3x^2}{\sqrt{2x^3+1}} \\
&= \frac{3x^2}{2x^3+1}
\end{aligned}
$$

YOU CAN NOW ATTEMPT THE WORKSHEET "DIFFERENTIATION OF LOGARITHMIC FUNCTIONS".

HIGHER DERIVATIVES

Given $y = f(x)$, the **first derivative** is denoted by $\frac{dy}{dx}$ or $f'(x)$.

The first derivative can then also be differentiated to give the **second derivative**.
The second derivative is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$.

Similarly, the second derivative can then also be differentiated to give the **third derivative,** denoted by $\frac{d^3y}{dx^3}$ or $f'''(x)$.

The second derivative gives information about the nature of a stationary point, as we will see shortly.

Example 1

$$y = 2x^4 + x^3 - 3x^2 + 4x - 1$$

$$\frac{dy}{dx} = 8x^3 + 3x^2 - 6x + 4$$

$$\frac{d^2y}{dx^2} = 24x^2 + 6x - 6$$

$$\frac{d^3y}{dx^3} = 48x + 6$$

$$\frac{d^4y}{dx^4} = 48$$

$$\frac{d^5y}{dx^5} = 0$$

Example 2

$$f(x) = \sin 3x$$

$$f'(x) = 3\cos 3x$$

$$f''(x) = 3(-3\sin 3x) = -9\sin 3x$$

$$f'''(x) = -9(3\cos 3x) = -27\cos 3x$$

$$f^{(4)}(x) = -27(-3\sin 3x) = 81\sin 3x$$

Example 3

$y = 4e^{2x} + 2e^{-3x}$

$$\frac{dy}{dx} = 4 \cdot 2e^{2x} + 2 \cdot (-3e^{-3x})$$
$$= 8e^{2x} - 6e^{-3x}$$

$$\frac{d^2y}{dx^2} = 8 \cdot 2e^{2x} - 6 \cdot (-3e^{-3x})$$
$$= 16e^{2x} + 18e^{-3x}$$

Example 4

$f(x) = (2x+1)^5$

$$f'(x) = 5(2x+1)^4 \cdot 2$$
$$= 10(2x+1)^4$$

$$f''(x) = 40(2x+1)^3 \cdot 2$$
$$= 80(2x+1)^3$$

Worked Example 5

Given $y = e^x \cos x$, obtain the second derivative $\frac{d^2y}{dx^2}$ in its simplest form.

Solution

$y = e^x \cos x$

This must be differentiated using the **product rule**.

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(e^x)$$
$$= e^x \cdot (-\sin x) + \cos x \cdot e^x$$
$$= e^x(\cos x - \sin x)$$

The first derivative can be differentiated again using the **product rule**.

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx}(\cos x - \sin x) + (\cos x - \sin x) \cdot \frac{d}{dx}(e^x)$$
$$= e^x(-\sin x - \cos x) + (\cos x - \sin x)e^x$$
$$= -e^x \sin x - e^x \cos x + e^x \cos x - e^x \sin x$$
$$= -2e^x \sin x$$

STATIONARY POINTS AND PONTS OF INFLEXION

The second derivative can be used to investigate the nature of stationary points on the curve $y = f(x)$.

(1) If $\frac{d^2y}{dx^2} > 0$ at a stationary point, the stationary point is a **minimum turning point**.

(2) If $\frac{d^2y}{dx^2} < 0$ at a stationary point, the stationary point is a **maximum turning point**.

(3) If $\frac{d^2y}{dx^2} = 0$ at a stationary point, the second derivative gives **no information** about the nature of the stationary point. The nature of the stationary point must be investigated by using a nature table.

$\frac{d^2y}{dx^2} > 0$	$\Rightarrow$	minimum
$\frac{d^2y}{dx^2} < 0$	$\Rightarrow$	maximum
$\frac{d^2y}{dx^2} = 0$	$\Rightarrow$	no information

If the second derivative of a function can be found easily, it is usually more efficient to use the value of the second derivative to determine the nature of a stationary point. If the second derivative cannot be found easily, a nature table should be used.

Worked Example

Determine the coordinates and nature of the stationary points on the curve $y = 2x^3 - 3x^2 - 12x$.

Solution

$$y = 2x^3 - 3x^2 - 12x \quad \Rightarrow \quad \frac{dy}{dx} = 6x^2 - 6x - 12$$

At a stationary point, $\frac{dy}{dx} = 0 \Rightarrow \quad 6x^2 - 6x - 12 = 0$

$$\Rightarrow \quad 6(x^2 - x - 2) = 0$$

$$\Rightarrow \quad 6(x-2)(x+1) = 0$$

$$\Rightarrow \quad x = 2 \text{ or } x = -1$$

When $x = 2$, $y = -20$ $\quad \Rightarrow \quad$ $(2, -20)$ is a stationary point

When $x = -1$, $y = 7$ $\quad \Rightarrow \quad$ $(-1, 7)$ is a stationary point

$$\frac{d^2y}{dx^2} = 12x - 6$$

When $x = 2$, $\frac{d^2y}{dx^2} = 12 \times 2 - 6 = 18 > 0 \quad \Rightarrow \quad$ $(2, -20)$ is a minimum turning point

When $x = -1$, $\frac{d^2y}{dx^2} = 12 \times (-1) - 6 = -18 < 0 \Rightarrow$ $(-1, 7)$ is a maximum turning point

In Advanced Higher, points of inflexion occur when the second derivative is zero.

At a point of inflexion, $\frac{d^2y}{dx^2} = 0$.

Worked Example 1

Find the coordinates of the points of inflexion on the curve $y = x^4 - 6x^2$.

Solution

$$y = x^4 - 6x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 4x^3 - 12x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 12x^2 - 12$$

At a point of inflexion, $\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad 12x^2 - 12 = 0$

$$\Rightarrow \quad 12x^2 = 12$$

$$\Rightarrow \quad x^2 = 1$$

$$\Rightarrow \quad x = \pm 1$$

When $x = 1$, $y = -5 \Rightarrow$ (1, –5) is a point of inflexion

When $x = -1$, $y = -5 \Rightarrow$ (–1, 5) is a point of inflexion

Worked Example 2

Show that there are no points of inflexion on the curve $y = \frac{x-3}{x+2}$, $x \neq -2$.

Solution

$$y = \frac{x-3}{x+2}$$

This must be differentiated using the **quotient rule**.

$$\frac{dy}{dx} = \frac{(x+2)\cdot 1 - (x-3)\cdot 1}{(x+2)^2} = \frac{x+2-x+3}{(x+2)^2} = \frac{5}{(x+2)^2}$$

The second derivative can be found by writing $\frac{dy}{dx}$ as $5(x+2)^{-2}$ and using the **chain rule**.

$$\frac{dy}{dx} = 5(x+2)^{-2} \qquad \Rightarrow \qquad \frac{d^2y}{dx^2} = -10(x+2)^{-3}\cdot 1 = -\frac{10}{(x+2)^3}$$

At a point of inflexion, $\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad -\frac{10}{(x+2)^3} = 0 \qquad [\times (x+2)^3]$

$$\Rightarrow \quad -10 = 0 \quad ???$$

This is a contradiction and means that there are no points of inflexion on the curve.

[Alternatively, the form $\frac{d^2y}{dx^2} = -\frac{10}{(x+2)^3}$ shows that $\frac{d^2y}{dx^2} > 0$ when $x < -2$ and $\frac{d^2y}{dx^2} < 0$ when $x > -2$. Hence $\frac{d^2y}{dx^2} \neq 0$ for any value of x ($x \neq -2$) and there are no point of inflexion on the curve.]

YOU CAN NOW ATTEMPT THE WORKSHEET "HIGHER DERIVATIVES".