BASIC INTEGRATION

USE OF STANDARD INTEGRALS

Recall the following standard integrals from Higher:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \qquad n \neq -1$$

$$\int (ax+b)^n\,dx = \frac{1}{a}\cdot\frac{1}{n+1}(ax+b)^{n+1} + C, \qquad n \neq -1$$

Example 1

$$\begin{aligned}
\int \frac{3x^2-1}{\sqrt{x}}\,dx &= \int \frac{3x^2-1}{x^{\frac{1}{2}}}\,dx \\
&= \int \left(\frac{3x^2}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}}\right)dx \\
&= \int \left(3x^{\frac{3}{2}} - x^{-\frac{1}{2}}\right)dx \\
&= 3\cdot\frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + C \\
&= \frac{6}{5}x^{\frac{5}{2}} - 2\sqrt{x} + C
\end{aligned}$$

Example 2

$$\begin{aligned}
\int (2x+3)^5\,dx &= \frac{1}{2}\cdot\frac{1}{6}(2x+3)^6 + C \\
&= \frac{1}{12}(2x+3)^6 + C
\end{aligned}$$

Example 3

$$\int \frac{2}{(3x+1)^5}\,dx = \int 2(3x+1)^{-5}\,dx$$

$$= 2 \cdot \frac{1}{3} \cdot \frac{1}{(-4)}(3x+1)^{-4} + C$$

$$= -\frac{1}{6}(3x+1)^{-4} + C$$

$$= -\frac{1}{6(3x+1)^4} + C$$

Example 4

$$\int_0^2 \sqrt{4x+1}\,dx = \int_0^2 (4x+1)^{\frac{1}{2}}\,dx$$

$$= \left[\frac{1}{4} \cdot \frac{2}{3}(4x+1)^{\frac{3}{2}}\right]_0^2$$

$$= \left[\frac{1}{6}(4x+1)^{\frac{3}{2}}\right]_0^2$$

$$= \left[\frac{1}{6} \times 9^{\frac{3}{2}}\right] - \left[\frac{1}{6} \times 1^{\frac{3}{2}}\right]$$

$$= 4\frac{1}{2} - \frac{1}{6}$$

$$= 4\frac{1}{3}$$

YOU CAN NOW ATTEMPT QUESTIONS 1 AND 2 OF THE WORKSHEET "INTEGRATION: USE OF STANDARD INTEGRALS".

INTEGRATION OF EXPONENTIAL FUNCTIONS

$$\int e^x dx = e^x + C$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

Note that the second standard integral only applies when integrating the exponential of a **linear function**. It cannot be used to find integrals such as $\int e^{x^2} dx$.

Example 1

$$\begin{aligned}\int (6e^{3x} + 4e^{-x})dx &= 6 \cdot \frac{1}{3} e^{3x} + 4 \cdot \frac{1}{(-1)} e^{-x} + C \\ &= 2e^{3x} - 4e^{-x} + C\end{aligned}$$

Example 2

$$\begin{aligned}\int \frac{8}{e^{2x}} dx &= \int 8e^{-2x} dx \\ &= 8 \cdot \frac{1}{(-2)} e^{-2x} + C \\ &= -4e^{-2x} + C\end{aligned}$$

Example 3

$$\begin{aligned}\int \left(e^x + \frac{1}{e^x}\right)^2 dx &= \int \left(e^x + \frac{1}{e^x}\right)\left(e^x + \frac{1}{e^x}\right) dx \\ &= \int \left(e^{2x} + 2 + \frac{1}{e^{2x}}\right) dx \\ &= \int (e^{2x} + 2 + e^{-2x}) dx \\ &= \frac{1}{2} e^{2x} + 2x + \frac{1}{(-2)} e^{-2x} + C \\ &= \frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} + C\end{aligned}$$

Worked Example 4

Find the **exact** value of the definite integral $\int_0^1 \frac{(e^{2x}-1)^2}{e^x}\,dx$.

Solution

$$\int_0^1 \frac{(e^{2x}-1)^2}{e^x}\,dx = \int_0^1 \frac{(e^{2x}-1)(e^{2x}-1)}{e^x}\,dx$$

$$= \int_0^1 \frac{e^{4x}-2e^{2x}+1}{e^x}\,dx$$

$$= \int_0^1 \left(\frac{e^{4x}}{e^x} - \frac{2e^{2x}}{e^x} + \frac{1}{e^x}\right)dx \qquad \text{[this step can be omitted with practice]}$$

$$= \int_0^1 (e^{3x} - 2e^x + e^{-x})\,dx$$

$$= \left[\frac{1}{3}e^{3x} - 2e^x + \frac{1}{(-1)}e^{-x}\right]_0^1 \qquad \text{[this step can be omitted with practice]}$$

$$= \left[\frac{1}{3}e^{3x} - 2e^x - e^{-x}\right]_0^1$$

$$= \left[\frac{1}{3}e^3 - 2e^1 - e^{-1}\right] - \left[\frac{1}{3}e^0 - 2e^0 - e^0\right]$$

$$= \frac{1}{3}e^3 - 2e - e^{-1} - \frac{1}{3} + 2 + 1 \qquad \text{[note that } e^0 = 1\text{]}$$

$$= \frac{1}{3}e^3 - 2e - e^{-1} + \frac{8}{3}$$

YOU CAN NOW ATTEMPT QUESTIONS 3, 4 AND 5 OF THE WORKSHEET "INTEGRATION: USE OF STANDARD INTEGRALS".

USE OF LOGARITHMS IN INTEGRATION

Recall that $\frac{d}{dx}\ln x = \frac{1}{x}$, $x > 0$. The condition "$x > 0$" is necessary as $\ln x$ is only defined when $x > 0$.

This means that $\int \frac{1}{x}\,dx = \ln x + C$, $x > 0$.

This result must be adapted slightly to find $\int \frac{1}{x}\,dx$ when $x < 0$.

The **magnitude** of a real number x is denoted by $|x|$ and is the positive numerical value of x, regardless of whether x itself is positive or negative.

The function $|x|$ is therefore defined as follows:

$$|x| = x \text{ if } x \geq 0$$
$$|x| = -x \text{ if } x < 0$$

Hence $|2| = 2$, $|-3| = 3$, and so on.

It can be shown that $\int \frac{1}{x}\,dx = \ln|x| + C$ for **all** non-zero real values of x.

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

The magnitude signs can be omitted in practice if the logarithm of a positive number is involved.

Note that the second standard integral only applies when integrating the reciprocal of a **linear** function. It cannot be used to find integrals such as $\int \frac{1}{x^2+1}\,dx$.

Example 1

$$\begin{aligned}\int \frac{8}{2x+1}\,dx &= 8 \cdot \frac{1}{2}\ln|2x+1| + C \\ &= 4\ln|2x+1| + C\end{aligned}$$

Example 2

$$\int\left(\frac{2}{x}+\frac{4}{6x+1}-\frac{6}{1-3x}\right)dx$$
$$=2\ln|x|+4\cdot\frac{1}{6}\ln|6x+1|-6\cdot\frac{1}{(-3)}\ln|1-3x|+C$$
$$=2\ln|x|+\frac{2}{3}\ln|6x+1|+2\ln|1-3x|+C$$

Worked Example 3

Show that $\int_2^{14}\frac{1}{2x-1}dx=\ln 3$.

Solution

$$\begin{aligned}\int_2^{14}\frac{1}{2x-1}dx &= \left[\frac{1}{2}\ln(2x-1)\right]_2^{14}\\ &=\left[\frac{1}{2}\ln 27\right]-\left[\frac{1}{2}\ln 3\right]\\ &=\frac{1}{2}(\ln 27-\ln 3)\\ &=\frac{1}{2}\ln\left(\frac{27}{3}\right)\\ &=\frac{1}{2}\ln 9\\ &=\ln(9^{\frac{1}{2}})\\ &=\ln 3\end{aligned}$$

[There are other ways of manipulating the logarithms in this question.
For example, $\frac{1}{2}\ln 27-\frac{1}{2}\ln 3=\frac{1}{2}\ln(3^3)-\frac{1}{2}\ln 3$
$=\frac{3}{2}\ln 3-\frac{1}{2}\ln 3$
$=\ln 3$.]

Note that there was no need to include the magnitude signs in this question as logarithms of positive numbers were involved.

YOU CAN NOW ATTEMPT QUESTIONS 6, 7 AND 8 OF THE WORKSHEET "INTEGRATION: USE OF STANDARD INTEGRALS".

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

You must learn the following standard integrals:

$$\int \sin x dx = -\cos x + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C$$

These standard integrals can be extended as follows:

$$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$$

The standard integrals above are the most important standard integrals for trigonometric functions, however there are also some other standard integrals which are occasionally used.

For example, we know that $\frac{d}{dx}\sec x = \sec x \tan x$, hence $\int \sec x \tan x dx = \sec x + C$.

By extension, $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + C$.

Example 1

$$\begin{aligned}\int (3\cos x - 4\sin 2x)dx &= 3\sin x - 4\left(-\frac{1}{2}\cos 2x\right) + C \\ &= 3\sin x + 2\cos 2x + C\end{aligned}$$

Example 2

$$\int 6\sec^2 4x\,dx = 6\cdot\frac{1}{4}\tan 4x + C$$
$$= \frac{3}{2}\tan 4x + C$$

Worked Example 3

Show that $\int_0^{\frac{\pi}{6}}(\sin 2x + \sec^2 x)dx = \frac{4\sqrt{3}+3}{12}$.

Solution

$$\int_0^{\frac{\pi}{6}}(\sin 2x + \sec^2 x)dx = \left[-\frac{1}{2}\cos 2x + \tan x\right]_0^{\frac{\pi}{6}}$$
$$= \left[-\frac{1}{2}\cos\left(2\times\frac{\pi}{6}\right) + \tan\frac{\pi}{6}\right] - \left[-\frac{1}{2}\cos(2\times 0) + \tan 0\right]$$
$$= -\frac{1}{2}\cos\frac{\pi}{3} + \tan\frac{\pi}{6} + \frac{1}{2}\cos 0 - \tan 0$$
$$= -\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{\sqrt{3}} + \frac{1}{2}\cdot 1 - 0$$
$$= -\frac{1}{4} + \frac{1}{\sqrt{3}} + \frac{1}{2}$$
$$= \frac{1}{4} + \frac{1}{\sqrt{3}}$$
$$= \frac{1}{4} + \frac{\sqrt{3}}{3} \quad \text{[note that } \frac{1}{\sqrt{3}} = \frac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{\sqrt{3}}{3}\text{]}$$
$$= \frac{3}{12} + \frac{4\sqrt{3}}{12}$$
$$= \frac{4\sqrt{3}+3}{12}$$

YOU CAN NOW ATTEMPT QUESTIONS 9, 10 AND 11 OF THE WORKSHEET "INTEGRATION: USE OF STANDARD INTEGRALS".

INTEGRATION BY SUBSTITUTION

Integration by substitution is useful when the derivative of one part of the integrand is related to another part of the integrand.

Integration by substitution involves rewriting the entire integral (including the "dx" and any limits) in terms of another variable before integrating. Note that the answer for an indefinite integral must always be expressed in terms of the original variable.

Worked Example 1

Use the substitution $u = x^2 + 1$ to find $\int x(x^2+1)^5\,dx$.

Solution

$$I = \int x(x^2+1)^5\,dx$$

$$\begin{aligned} u = x^2 + 1 \quad &\Rightarrow \quad \frac{du}{dx} = 2x \\ &\Rightarrow \quad du = 2x\,dx \\ &\Rightarrow \quad \frac{1}{2}du = x\,dx \end{aligned}$$

This means that the term xdx in the integrand can be replaced by $\frac{1}{2}du$.

Also, the term $(x^2+1)^5$ can be replaced by u^5.

$$\begin{aligned} I = \int x(x^2+1)^5\,dx &= \int u^5 \cdot \frac{1}{2}du \\ &= \int \frac{1}{2}u^5\,du \\ &= \frac{1}{2}\cdot\frac{1}{6}u^6 + C \\ &= \frac{1}{12}u^{12} + C \\ &= \frac{1}{12}(x^2+1)^{12} + C \end{aligned}$$

Worked Example 2

Use the substitution $u = 1 - x^3$ to find $\int x^2 \sqrt{1 - x^3}\, dx$.

Solution

$$I = \int x^2 \sqrt{1 - x^3}\, dx$$

$$u = 1 - x^3 \quad \Rightarrow \quad \frac{du}{dx} = -3x^2$$

$$\Rightarrow \quad du = -3x^2 dx$$

$$\Rightarrow \quad -\frac{1}{3} du = x^2 dx$$

$$I = \int x^2 \sqrt{1 - x^3}\, dx = \int \sqrt{u} \cdot \left(-\frac{1}{3} du \right)$$

$$= \int -\frac{1}{3} u^{\frac{1}{2}} du$$

$$= -\frac{1}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= -\frac{2}{9} u^{\frac{3}{2}} + C$$

$$= -\frac{2}{9} (1 - x^3)^{\frac{3}{2}} + C$$

Worked Example 3

Use the substitution $u = \sin x$ to find $\int \sin^3 x \cos x dx$.

Solution

$$I = \int \sin^3 x \cos x dx$$

$$u = \sin x \quad \Rightarrow \quad \frac{du}{dx} = \cos x$$

$$\Rightarrow \quad du = \cos x dx$$

$$\begin{aligned} I = \int \sin^3 x \cos x dx &= \int u^3 du \\ &= \frac{1}{4} u^4 + C \\ &= \frac{1}{4} \sin^4 x + C \end{aligned}$$

YOU CAN NOW ATTEMPT QUESTION 1 OF THE WORKSHEET "INTEGRATION BY SUBSTITUTION".

Worked Example 4

Use the substitution $u = 3x^2 + 1$ to find $\int \frac{x}{3x^2+1}\,dx$.

Solution

$$I = \int \frac{x}{3x^2+1}\,dx$$

$$u = 3x^2+1 \quad \Rightarrow \quad \frac{du}{dx} = 6x$$

$$\Rightarrow \quad du = 6x\,dx$$

$$\Rightarrow \quad \frac{1}{6}du = x\,dx$$

$$I = \int \frac{x}{3x^2+1}\,dx = \int \frac{1}{3x^2+1} \cdot x\,dx$$

$$= \int \frac{1}{u} \cdot \frac{1}{6}\,du$$

$$= \frac{1}{6}\ln|u| + C$$

$$= \frac{1}{6}\ln|3x^2+1| + C$$

Worked Example 5

Use the substitution $u = 1 + \cos 2x$ to find $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\, dx$.

Solution

$$I = \int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\, dx$$

$$u = 1 + \cos 2x \Rightarrow \quad \frac{du}{dx} = -2\sin 2x$$

$$\Rightarrow \quad du = -2\sin 2x dx$$

$$\Rightarrow \quad -\frac{1}{2} du = \sin 2x dx$$

$$\begin{aligned}
I = \int \frac{\sin 2x}{\sqrt{1+\cos 2x}}\, dx &= \int \frac{1}{\sqrt{1+\cos 2x}} \cdot \sin 2x dx \\
&= \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{2} du\right) \\
&= \int -\frac{1}{2} u^{-\frac{1}{2}} du \\
&= -\frac{1}{2} \cdot 2u^{\frac{1}{2}} + C \\
&= -\sqrt{u} + C \\
&= -\sqrt{1+\cos 2x} + C
\end{aligned}$$

Worked Example 6

Use the substitution $u = x^3 + 1$ to find $\int \frac{x^2}{(x^3+1)^5} dx$.

Solution

$$I = \int \frac{x^2}{(x^3+1)^5} dx$$

$$u = x^3 + 1 \quad \Rightarrow \quad \frac{du}{dx} = 3x^2$$

$$\Rightarrow \quad du = 3x^2 dx$$

$$\Rightarrow \quad \frac{1}{3} du = x^2 dx$$

$$\begin{aligned}
I = \int \frac{x^2}{(x^3+1)^5} dx &= \int \frac{1}{(x^3+1)^5} \cdot x^2 dx \\
&= \int \frac{1}{u^5} \cdot \frac{1}{3} du \\
&= \int \frac{1}{3} u^{-5} du \\
&= \frac{1}{3} \cdot \frac{1}{(-4)} u^{-4} + C \\
&= -\frac{1}{12} u^{-4} + C \\
&= -\frac{1}{12u^4} + C \\
&= -\frac{1}{12(x^3+1)^4} + C
\end{aligned}$$

YOU CAN NOW ATTEMPT QUESTIONS 2 AND 3 OF THE WORKSHEET "INTEGRATION BY SUBSTITUTION".

Worked Example 7

Use the substitution $u = x + 1$ to find $\int x(x+1)^3\,dx$.

Solution

$$I = \int x(x+1)^3\,dx$$

$$u = x + 1 \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\Rightarrow \quad du = dx$$

$$u = x + 1 \quad \Rightarrow \quad x + 1 = u$$

$$\Rightarrow \quad x = u - 1$$

$$\begin{aligned} I = \int x(x+1)^3\,dx &= \int (u-1)\cdot u^3 \cdot du \\ &= \int (u^4 - u^3)\,du \\ &= \frac{1}{5}u^5 - \frac{1}{4}u^4 + C \\ &= \frac{1}{5}(x+1)^5 - \frac{1}{4}(x+1)^4 + C \end{aligned}$$

Worked Example 8

Use the substitution $u = x - 2$ to find $\int \frac{x+1}{\sqrt{x-2}}\,dx$.

Solution

$$I = \int \frac{x+1}{\sqrt{x-2}}\,dx$$

$$u = x - 2 \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\Rightarrow \quad du = dx$$

$$u = x - 2 \quad \Rightarrow \quad x - 2 = u$$

$$\Rightarrow \quad x = u + 2$$

$$\Rightarrow \quad x + 1 = u + 3$$

$$I = \int \frac{x+1}{\sqrt{x-2}}\,dx = \int \frac{u+3}{\sqrt{u}}\,du$$

$$= \int \frac{u+3}{u^{\frac{1}{2}}}\,du$$

$$= \int \left(\frac{u^1}{u^{\frac{1}{2}}} + \frac{3}{u^{\frac{1}{2}}} \right) du$$

$$= \int (u^{\frac{1}{2}} + 3u^{-\frac{1}{2}})\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + 3 \cdot 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}u^{\frac{3}{2}} + 6\sqrt{u} + C$$

$$= \frac{2}{3}(x-2)^{\frac{3}{2}} + 6\sqrt{x-2} + C$$

Worked Example 9

Use the substitution $u = x^2 + 1$ to find $\int x^3\sqrt{x^2+1}\,dx$.

Solution

$$I = \int x^3\sqrt{x^2+1}\,dx$$

$$\begin{aligned} u = x^2 + 1 \quad &\Rightarrow \quad \frac{du}{dx} = 2x \\ &\Rightarrow \quad du = 2x\,dx \\ &\Rightarrow \quad \frac{1}{2}du = x\,dx \end{aligned}$$

The term $\sqrt{x^2+1}$ in the integrand can be replaced by $\sqrt{u}$ and the term xdx can be replaced by $\frac{1}{2}du$.

We must also replace the remaining term x^2 in the integrand.

$$\begin{aligned} u = x^2 + 1 \quad &\Rightarrow \quad x^2 + 1 = u \\ &\Rightarrow \quad x^2 = u - 1 \end{aligned}$$

$$\begin{aligned} I = \int x^3\sqrt{x^2+1}\,dx &= \int x^2 \cdot \sqrt{x^2+1} \cdot x\,dx \\ &= \int (u-1)\cdot\sqrt{u}\cdot\frac{1}{2}du \\ &= \int \frac{1}{2}u^{\frac{1}{2}}(u-1)\,du \\ &= \int\left(\frac{1}{2}u^{\frac{3}{2}} - \frac{1}{2}u^{\frac{1}{2}}\right)du \\ &= \frac{1}{2}\cdot\frac{2}{5}u^{\frac{5}{2}} - \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}} + C \\ &= \frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C \end{aligned}$$

YOU CAN NOW ATTEMPT QUESTIONS 4, 5, 6 AND 7 OF THE WORKSHEET "INTEGRATION BY SUBSTITUTION".

When using integration by substitution to evaluate a definite integral, the limits of the definite integral must be expressed in terms of the new variable.

Worked Example 10

Use the substitution $u = x^4 + 1$ to evaluate the definite integral $\int_0^1 x^3(x^4+1)^3 dx$.

Solution

$$I = \int_0^1 x^3(x^4+1)^3 dx$$

$$\begin{aligned} u = x^4 + 1 \quad &\Rightarrow \quad \frac{du}{dx} = 4x^3 \\ &\Rightarrow \quad du = 4x^3 dx \\ &\Rightarrow \quad \frac{1}{4} du = x^3 dx \end{aligned}$$

When $x = 0$: $u = 0^4 + 1 = 1$
When $x = 1$: $u = 1^4 + 1 = 2$

$$\begin{aligned} I &= \int_0^1 x^3(x^4+1)^3 dx \\ &= \int_1^2 u^3 \cdot \frac{1}{4} du \\ &= \left[\frac{1}{4}\cdot\frac{1}{4}u^4\right]_1^2 \\ &= \left[\frac{1}{16}u^4\right]_1^2 \\ &= \left[\frac{1}{16}\times 2^4\right] - \left[\frac{1}{16}\times 1^4\right] \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16} \end{aligned}$$

Worked Example 11

Use the substitution $u = \cos x$ to find the exact value of the definite integral $\int_0^{\frac{\pi}{3}} \tan x dx$.

Solution

$$I = \int_0^{\frac{\pi}{3}} \tan x dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$$

$$u = \cos x \quad \Rightarrow \quad \frac{du}{dx} = -\sin x$$

$$\Rightarrow \quad du = -\sin x dx$$

$$\Rightarrow \quad -du = \sin x dx$$

When $x = 0$: $u = \cos 0 = 1$

When $x = \frac{\pi}{3}$: $u = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\begin{aligned}
I = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx &= \int_0^{\frac{\pi}{3}} \frac{1}{\cos x} \cdot \sin x dx \\
&= \int_1^{\frac{1}{2}} \frac{1}{u} \cdot (-du) \\
&= \left[-\ln u\right]_1^{\frac{1}{2}} \\
&= \left[-\ln\left(\frac{1}{2}\right)\right] - \left[-\ln 1\right] \\
&= -\ln\left(\frac{1}{2}\right) \qquad [\text{since } \ln 1 = 0] \\
&= -\ln(2^{-1}) \\
&= \ln 2
\end{aligned}$$

[*or* $-\ln\left(\frac{1}{2}\right) = -\{\ln 1 - \ln 2\} = -\ln 1 + \ln 2 = \ln 2$, since $\ln 1 = 0$.]

Worked Example 12

Use the substitution $u = x - 1$ to evaluate the definite integral $\int_2^5 x\sqrt{x-1}\,dx$.

Solution

$$I = \int_2^5 x\sqrt{x-1}\,dx$$

$$u = x - 1 \quad \Rightarrow \quad \frac{du}{dx} = 1$$
$$\Rightarrow \quad du = dx$$

$$u = x - 1 \quad \Rightarrow \quad x - 1 = u$$
$$\Rightarrow \quad x = u + 1$$

When $x = 2$: $u = 2 - 1 = 1$
When $x = 5$: $u = 5 - 1 = 4$

$$I = \int_2^5 x\sqrt{x-1}\,dx = \int_1^4 (u+1)\cdot\sqrt{u}\cdot du$$
$$= \int_1^4 u^{\frac{1}{2}}(u+1)\,du$$
$$= \int_1^4 (u^{\frac{3}{2}} + u^{\frac{1}{2}})\,du$$
$$= \left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_1^4$$
$$= \left[\frac{2}{5}\times 4^{\frac{5}{2}} + \frac{2}{3}\times 4^{\frac{3}{2}}\right] - \left[\frac{2}{5}\times 1^{\frac{5}{2}} + \frac{2}{3}\times 1^{\frac{3}{2}}\right]$$
$$= 18\frac{2}{15} - 1\frac{1}{15}$$
$$= 17\frac{1}{15}$$

Worked Example 13

Use the substitution $u = \sin x$ to evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x dx$.

Solution

$$I = \int_0^{\frac{\pi}{2}} \cos^3 x dx$$

$$u = \sin x \quad \Rightarrow \quad \frac{du}{dx} = \cos x$$
$$\Rightarrow \quad du = \cos x dx$$

The term $\cos x dx$ in the integrand can be replaced by du.
We must also replace the remaining term $\cos^2 x$ in the integrand.

$$\cos^2 x = 1 - \sin^2 x = 1 - u^2$$

When $x = 0$: $u = \sin 0 = 0$
When $x = \frac{\pi}{2}$: $u = \sin \frac{\pi}{2} = 1$

$$I = \int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \cos x dx$$
$$= \int_0^1 (1 - u^2) du$$
$$= \left[u - \frac{1}{3} u^3 \right]_0^1$$
$$= \left[1 - \frac{1}{3} \times 1^3 \right] - \left[0 - \frac{1}{3} \times 0^3 \right]$$
$$= \frac{2}{3}$$

YOU CAN NOW ATTEMPT QUESTIONS 8 AND 9 OF THE WORKSHEET "INTEGRATION BY SUBSTITUTION".

If the integrand contains a term of the form $\sqrt{a^2 - x^2}$, the substitution $x = a\sin\theta$ is often useful.

Worked Example 14

Use the substitution $x = 2\sin\theta$ to evaluate the definite integral $\int_0^1 \frac{x+1}{\sqrt{4-x^2}}dx$.

Solution

$$I = \int_0^1 \frac{x+1}{\sqrt{4-x^2}}dx$$

$$x = 2\sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = 2\cos\theta$$
$$\Rightarrow \quad dx = 2\cos\theta d\theta$$

$$\begin{aligned} 4 - x^2 &= 4 - (2\sin\theta)^2 \\ &= 4 - 4\sin^2\theta \\ &= 4(1 - \sin^2\theta) \\ &= 4\cos^2\theta \quad [\text{since } \cos^2\theta = 1 - \sin^2\theta] \end{aligned}$$

Hence $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$.

When $x = 0$: $0 = 2\sin\theta \quad \Rightarrow \quad \sin\theta = 0$
$\Rightarrow \quad \theta = 0$

When $x = 1$: $1 = 2\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{1}{2}$
$\Rightarrow \quad \theta = \frac{\pi}{6}$

$$\begin{aligned} I &= \int_0^1 \frac{x+1}{\sqrt{4-x^2}}dx \\ &= \int_0^{\frac{\pi}{6}} \frac{(2\sin\theta + 1)}{2\cos\theta} \cdot 2\cos\theta d\theta \\ &= \int_0^{\frac{\pi}{6}} (2\sin\theta + 1)d\theta \\ &= \left[-2\cos\theta + \theta\right]_0^{\frac{\pi}{6}} \\ &= \left[-2\cos\frac{\pi}{6} + \frac{\pi}{6}\right] - \left[-2\cos 0 + 0\right] \end{aligned}$$

$$= -2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} + 2(1)$$

$$= -\sqrt{3} + \frac{\pi}{6} + 2$$

YOU CAN NOW ATTEMPT QUESTIONS 10 AND 11 OF THE WORKSHEET "INTEGRATION BY SUBSTITUTION".

In some simple cases, you will be expected to decide on a suitable substitution yourself. Remember that integration by substitution is useful when the derivative of one part of the integrand is related to another part of the integrand. The following miscellaneous examples illustrate this.

Example 15

$$I = \int x\sqrt{x^2+1}\,dx$$

Note that the derivative of x^2+1 is related to x. This suggests that we should try the substitution $u = x^2+1$.

$$u = x^2+1 \quad \Rightarrow \quad \frac{du}{dx} = 2x$$
$$\Rightarrow \quad du = 2x\,dx$$
$$\Rightarrow \quad \frac{1}{2}du = x\,dx$$

$$I = \int x\sqrt{x^2+1}\,dx = \int \sqrt{u}\cdot\frac{1}{2}du$$
$$= \int \frac{1}{2}u^{\frac{1}{2}}du$$
$$= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C$$
$$= \frac{1}{3}u^{\frac{3}{2}} + C$$
$$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

Example 16

$$I = \int \frac{\sin x}{1+2\cos x}dx$$

Note that the derivative of $1+2\cos x$ is related to $\sin x$. This suggests that we should try the substitution $u = 1+2\cos x$.

$$u = 1+2\cos x \Rightarrow \quad \frac{du}{dx} = -2\sin x$$
$$\Rightarrow \quad du = -2\sin x\,dx$$
$$\Rightarrow \quad -\frac{1}{2}du = \sin x\,dx$$

$$I = \int \frac{\sin x}{1+2\cos x}\,dx = \int \frac{1}{1+2\cos x}\cdot \sin x dx$$

$$= \int \frac{1}{u}\cdot\left(-\frac{1}{2}du\right)$$

$$= -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|1+2\cos x| + C$$

Example 17

$$I = \int_0^3 \frac{x}{\sqrt{x+1}}\,dx$$

Try the substitution $u = x+1$.

$u = x+1 \quad \Rightarrow \quad \frac{du}{dx} = 1$

$\quad \Rightarrow \quad du = dx$

$u = x+1 \quad \Rightarrow \quad x+1 = u$

$\quad \Rightarrow \quad x = u-1$

When $x = 0$: $u = 0+1 = 1$
When $x = 3$: $u = 3+1 = 4$

$$I = \int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^4 \frac{u-1}{\sqrt{u}}du$$

$$= \int_1^4 \left(\frac{u^1}{u^{\frac{1}{2}}} - \frac{1}{u^{\frac{1}{2}}}\right)du$$

$$= \int_1^4 (u^{\frac{1}{2}} - u^{-\frac{1}{2}})du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$$

$$= \left[\frac{2}{3}\times 4^{\frac{3}{2}} - 2\times 4^{\frac{1}{2}}\right] - \left[\frac{2}{3}\times 1^{\frac{3}{2}} - 2\times 1^{\frac{1}{2}}\right]$$

$$= 1\frac{1}{3} - \left(-1\frac{1}{3}\right)$$

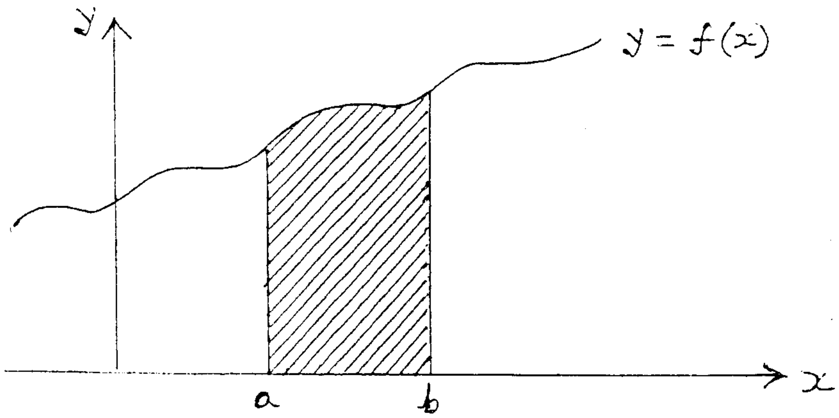
$$= 2\frac{2}{3}$$

YOU CAN NOW ATTEMPT QUESTIONS 12, 13 AND 14 OF THE WORKSHEET "INTEGRATION BY SUBSTITUTION".

APPLICATIONS OF INTEGRATION

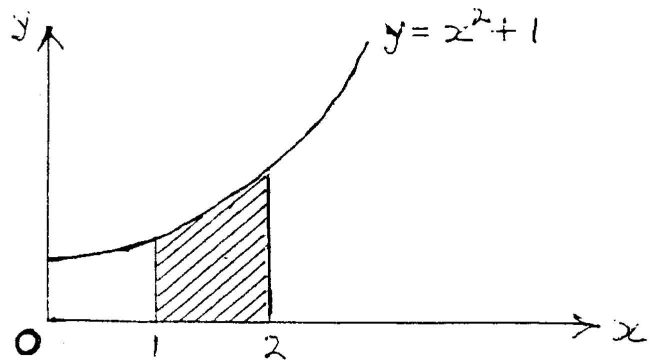
THE AREA BETWEEN A CURVE AND THE x-AXIS

Consider in general the area enclosed between the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$.



$$\text{Area} = \int_a^b y\,dx$$

Example



$$\text{Area} = \int_1^2 y\,dx = \int_1^2 (x^2 + 1)\,dx$$

$$= \left[\frac{1}{3}x^3 + x\right]_1^2$$

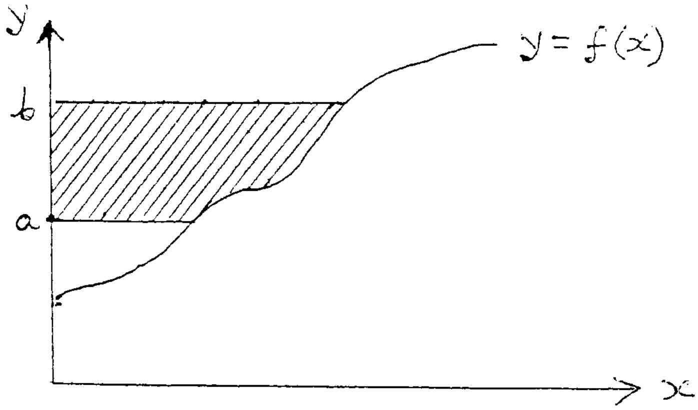
$$= \left[\frac{1}{3} \times 2^3 + 2\right] - \left[\frac{1}{3} \times 1^3 + 1\right]$$

$$= 4\frac{2}{3} - 1\frac{1}{3}$$

$$= 3\frac{1}{3} \text{ square units}$$

THE AREA BETWEEN A CURVE AND THE y-AXIS

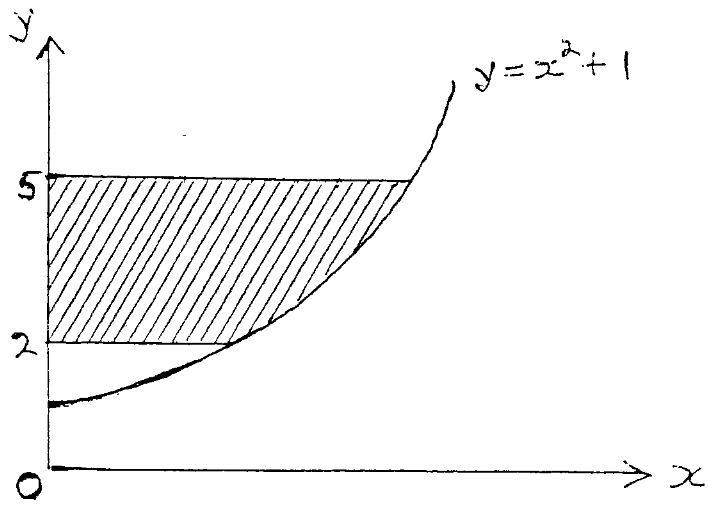
Now consider in general the area enclosed between the curve $y = f(x)$ and the y-axis between $y = a$ and $y = b$.



$$\text{Area} = \int_a^b x\,dy$$

Note that to evaluate this definite integral, x must be expressed in terms of y.

Example



$y = x^2 + 1 \quad \Rightarrow \quad x^2 + 1 = y$

$\Rightarrow \quad x^2 = y - 1$

$\Rightarrow \quad x = \sqrt{y - 1}$

[Note that $x \geq 0$ in the region concerned, therefore x is given by the positive square root.

$$\text{Area} = \int_2^5 x\,dy = \int_2^5 \sqrt{y-1}\,dy$$

$$= \int_2^5 (y-1)^{\frac{1}{2}}\,dy$$

$$= \left[\frac{2}{3}(y-1)^{\frac{3}{2}}\right]_2^5$$

$$= \left[\frac{2}{3} \times 4^{\frac{3}{2}}\right] - \left[\frac{2}{3} \times 1^{\frac{3}{2}}\right]$$

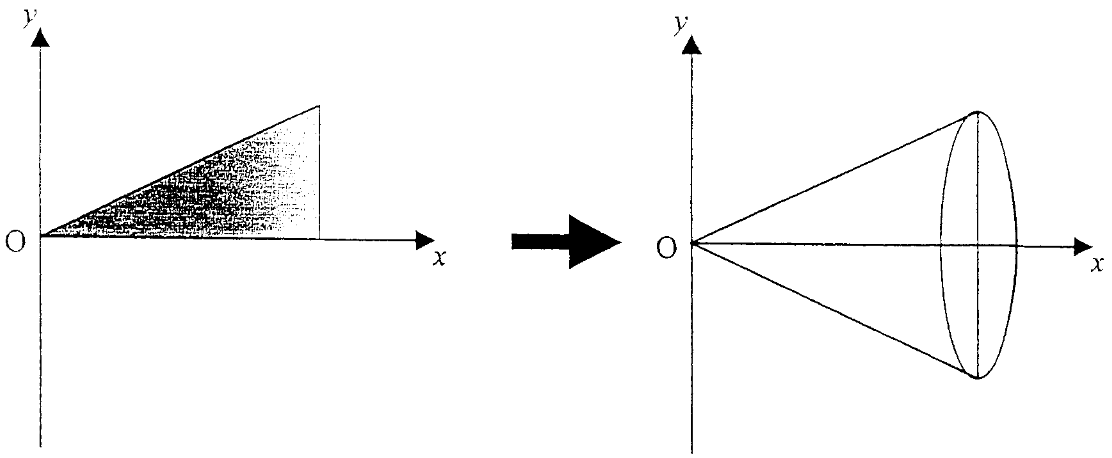
$$= 5\frac{1}{3} - \frac{2}{3}$$

$$= 4\frac{2}{3} \text{ square units}$$

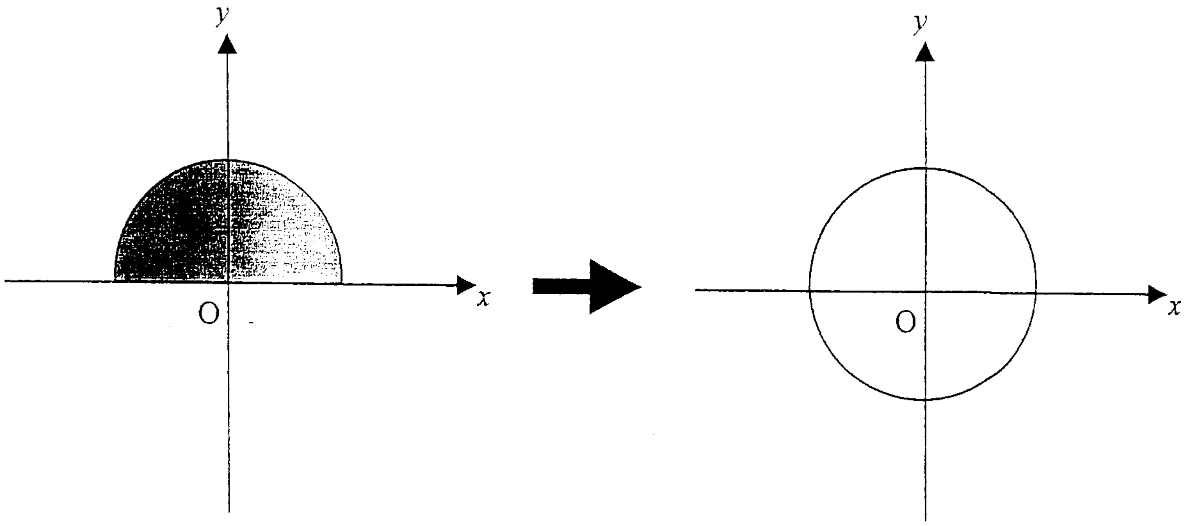
VOLUMES OF SOLIDS OF REVOLUTION

When a region in the x,y-plane is rotated through $360°$ about the x-axis, a solid of revolution is formed.

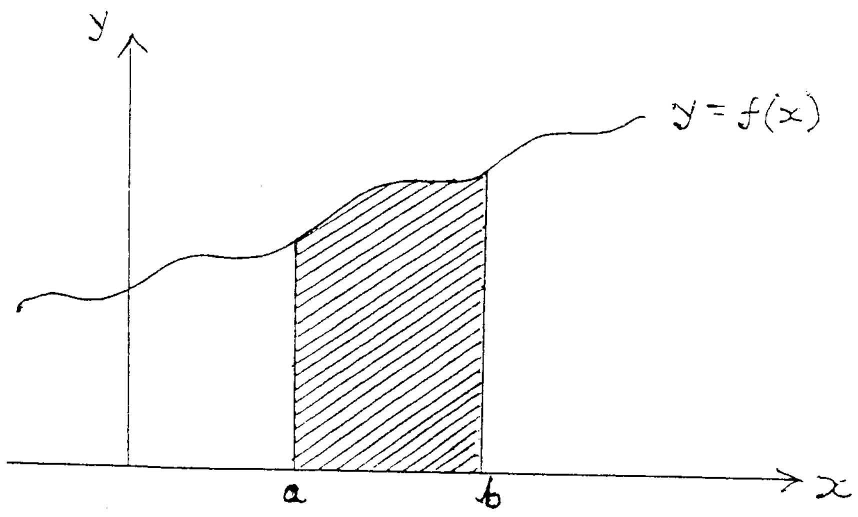
For example, when the shaded triangle below is rotated through $360°$ about the x-axis, a cone is formed.



When the shaded semi-circle below is rotated through $360°$ about the x-axis, a sphere is formed.



Suppose in general the region enclosed by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is rotated through $360°$ about the x-axis.

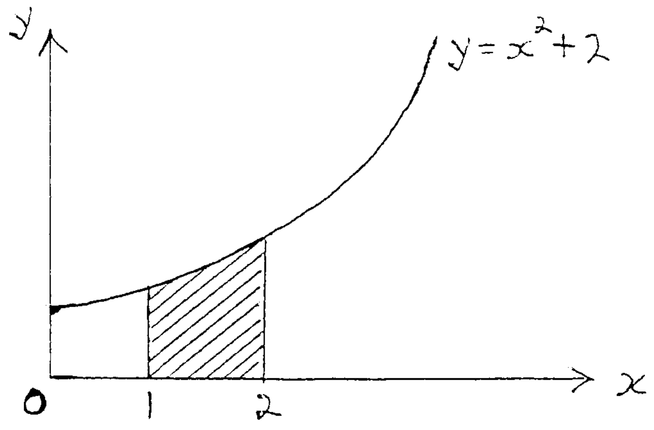


Then it can be shown that the volume of the solid of revolution formed is given by:

$$V = \pi \int_a^b y^2 \, dx$$

Worked Example

Find the volume of the solid formed when the shaded region below is rotated through $360°$ about the x-axis.



$$V = \pi \int_1^2 y^2 dx$$

$$\int_1^2 y^2 dx = \int_1^2 (x^2+2)^2 dx$$

$$= \int_1^2 (x^4 + 4x^2 + 4) dx$$

$$= \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x\right]_1^2$$

$$= \left[\frac{1}{5}\times 2^5 + \frac{4}{3}\times 2^3 + 4\times 2\right] - \left[\frac{1}{5}\times 1^5 + \frac{4}{3}\times 1^3 + 4\times 1\right]$$

$$= 25\frac{1}{15} - 5\frac{8}{15}$$

$$= 19\frac{8}{15}$$

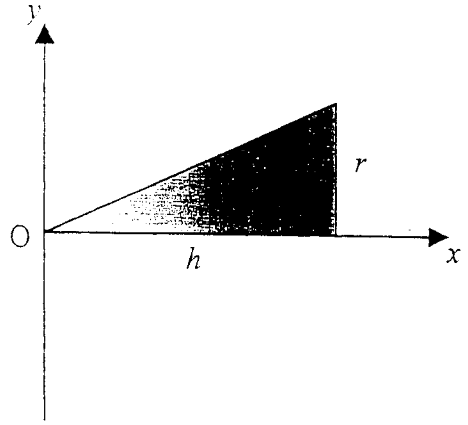
$$= \frac{293}{15}$$

$$V = \pi \times \frac{293}{15} = \frac{293\pi}{15} \text{ cubic units}$$

YOU CAN NOW ATTEMPT QUESTIONS 1 AND 2 OF THE WORKSHEET "VOLUMES OF SOLIDS OF REVOLUTION".

Optional Example 1

The formula for the volume of a cone can be found by considering a volume of solid of revolution.



When the shaded region above is rotated through $360°$ about the x-axis, a cone with radius r and height h is formed.

The straight line shown passes through the origin and has gradient $\frac{r}{h}$, hence the equation of the straight line is $y = \frac{r}{h}x$.

$$V = \pi \int_0^h y^2 dx$$

$$\int_0^h y^2 dx = \int_0^h \left(\frac{r}{h}x\right)^2 dx$$

$$= \int_0^h \frac{r^2}{h^2} x^2 dx$$

$$= \left[\frac{r^2}{h^2} \cdot \frac{1}{3} x^3\right]_0^h$$

[note that r and h are constants and the integration is performed with respect to x]

$$= \left[\frac{r^2}{3h^2} x^3\right]_0^h$$

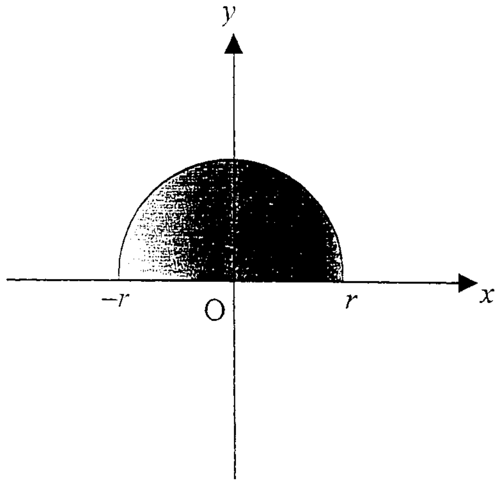
$$= \left[\frac{r^2}{3h^2} \times h^3\right] - \left[\frac{r^2}{3h^2} \times 0^3\right]$$

$$= \frac{r^2 h}{3}$$

Hence $V = \pi \times \frac{r^2 h}{3} = \frac{1}{3}\pi r^2 h$.

Optional Example 2

The formula for the volume of a sphere can also be found by considering the volume of a solid of revolution.



When the shaded region above is rotated through $360°$ about the x-axis, a sphere with radius r is formed.

The shaded region is a semicircle with centre O and radius r. The equation of the semi-circle is thus $x^2 + y^2 = r^2$ with $y \geq 0$. This equation can be expressed as $y^2 = r^2 - x^2$ with $y \geq 0$.

$$V = \pi \int_{-r}^{r} y^2 dx$$

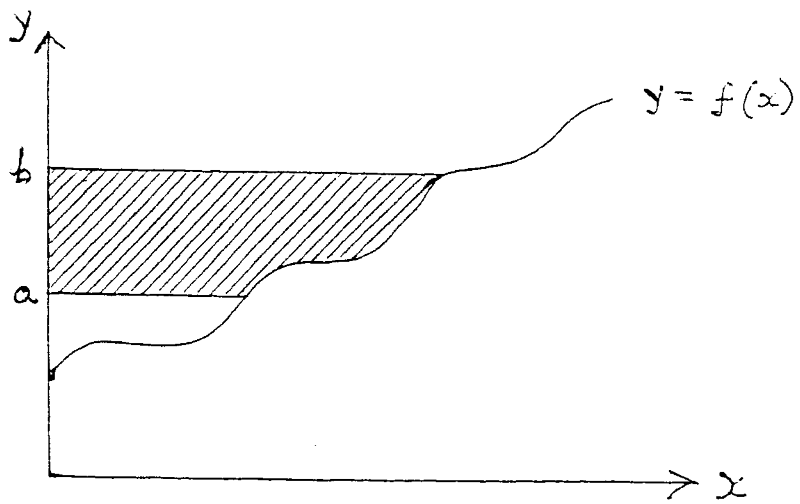
$$\begin{aligned}
\int_{-r}^{r} y^2 dx &= \int_{-r}^{r} (r^2 - x^2) dx \\
&= \left[r^2 x - \frac{1}{3} x^3 \right]_{-r}^{r} \\
&= \left[r^2 r - \frac{1}{3} r^3 \right] - \left[r^2(-r) - \frac{1}{3}(-r)^3 \right] \\
&= \left[r^3 - \frac{1}{3} r^3 \right] - \left[-r^3 + \frac{1}{3} r^3 \right] \\
&= r^3 - \frac{1}{3} r^3 + r^3 - \frac{1}{3} r^3 \\
&= \frac{4}{3} r^3
\end{aligned}$$

[note that r is a constant and the integration is performed with respect to x]

Hence $V = \pi \times \frac{4}{3} r^3 = \frac{4}{3} \pi r^3$.

A solid of revolution is also formed when a region in the x, y-plane is rotated through 360° about the y-axis.

Suppose in general the region enclosed by the curve $y = f(x)$ and the y-axis between $y = a$ and $y = b$ is rotated through 360° about the y-axis.



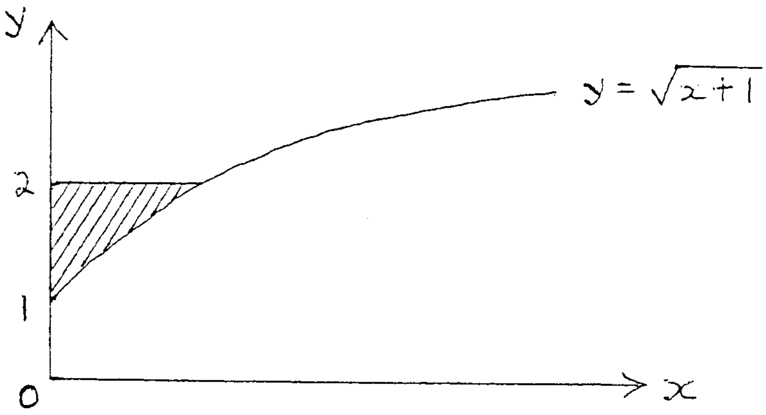
Then it can be shown that the volume of the solid of revolution formed is given by:

$$V = \pi \int_a^b x^2 \, dy$$

Note that to evaluate this definite integral, x^2 must be expressed in terms of y.

Worked Example

Find the volume of the solid formed when the shaded region below is rotated through 360° about the y-axis.



$$V = \pi \int_1^2 x^2\,dy$$

Now $y = \sqrt{x+1}$ $\Rightarrow$ $y^2 = x+1$

$\Rightarrow$ $x+1 = y^2$

$\Rightarrow$ $x = y^2 - 1$

$\Rightarrow$ $x^2 = (y^2-1)^2$

$$\int_1^2 x^2\,dy = \int_1^2 (y^2-1)^2\,dy$$

$$= \int_1^2 (y^4 - 2y^2 + 1)dy$$

$$= \left[\frac{1}{5}y^5 - \frac{2}{3}y^3 + y\right]_1^2$$

$$= \left[\frac{1}{5}\times 2^5 - \frac{2}{3}\times 2^3 + 2\right] - \left[\frac{1}{5}\times 1^5 - \frac{2}{3}\times 1^3 + 1\right]$$

$$= 3\frac{1}{15} - \frac{8}{15}$$

$$= 2\frac{8}{15}$$

$$= \frac{38}{15}$$

Hence $V = \pi \times \frac{38}{15} = \frac{38\pi}{15}$ cubic units.

YOU CAN NOW ATTEMPT QUESTION 3 OF THE WORKSHEET "VOLUMES OF SOLIDS OF REVOLUTION".