PROPERTIES OF FUNCTIONS

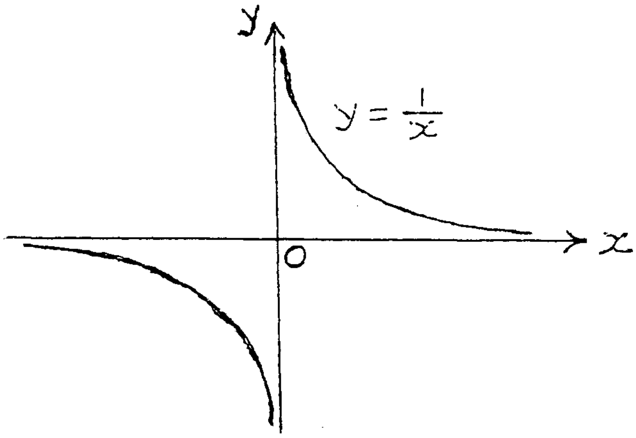
SKETCHING RATIONAL FUNCTIONS

A **rational function** is a function of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials in x.

Consider the graph of the simple rational function $y = \frac{1}{x}$, $x \neq 0$.

x	0·001	0·01	0·1	1	10	100	1000
y	1000	100	10	1	0·1	0·01	0·001

The behaviour of y for negative values of x can be investigated similarly.



The graph approaches the x-axis and y-axis but does not actually touch either axis. The x-axis and y-axis are known as **asymptotes**.

Note that the graph "jumps" suddenly at either side of $x = 0$. The graph is said to be discontinuous at $x = 0$.

General Notes

(1) Vertical asymptotes always occur when the denominator of the function equals zero.

(2) At either side of a vertical asymptote, $y \to \infty$ or $y \to -\infty$. The behaviour of a graph at either side of a vertical asymptote should always be investigated.

(3) Non-vertical asymptotes occur when $x \to \pm\infty$.

(4) Any points of intersection with the coordinate axes should be investigated.

(5) The coordinates and nature of any stationary points should be found when requested.

Worked Example 1

Sketch the graph of $y = \dfrac{1}{(x+2)(x-3)}$.

[You need not find the coordinates of any stationary points.]

Solution

y-axis: When $x = 0$, $y = \dfrac{1}{(2)(-3)} = -\dfrac{1}{6}$.

The curve cuts the y-axis at $\left(0, -\dfrac{1}{6}\right)$.

x-axis: When $y = 0$, $\dfrac{1}{(x+2)(x-3)} = 0 \quad \Rightarrow \quad 1 = 0$???

This means that the curve does not cut the x-axis.

Vertical Asymptotes: $(x+2)(x-3) = 0 \quad \Rightarrow \quad x = -2$ or $x = 3$

The behaviour of the curve at either side of these vertical asymptotes must be investigated. We know that $y \to \infty$ or $y \to -\infty$ before or after a vertical asymptote. The simplest way to investigate the behaviour is to calculate y for a value of x just before a vertical asymptote and for a value of y just after the vertical asymptote. If the calculated value of y is positive, it means that $y \to \infty$ and if the value of y is negative, it means that $y \to -\infty$.

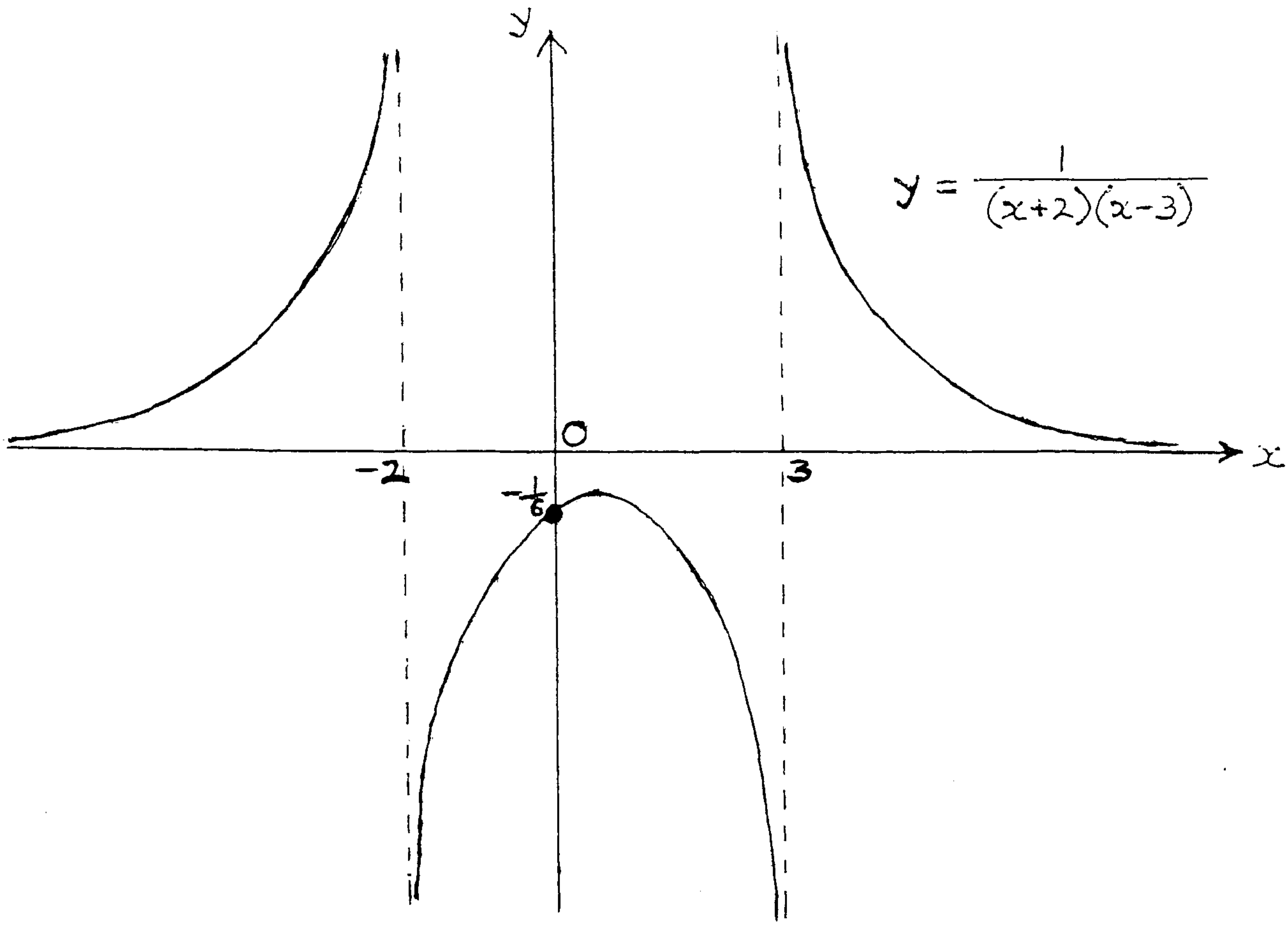
x	$-2{\cdot}1$	-2	$-1{\cdot}9$	$2{\cdot}9$	3	$3{\cdot}1$
y	$+\infty$		$-\infty$	$-\infty$		$+\infty$

Non-Vertical Asymptote:

$$y = \frac{1}{(x+2)(x-3)}$$

As $x \to \pm\infty$, $y \to 0$.

This means that $y = 0$ is a non-vertical asymptote.


y
O
x
-2
3
$-\frac{1}{6}$
$y = \frac{1}{(x+2)(x-3)}$

Worked Example 2

Sketch the graph of $y = \dfrac{x-3}{x^2+x-2}$.

[You need not find the coordinates of any stationary points.]

Solution

y-axis: When $x = 0$, $y = \dfrac{-3}{-2} = \dfrac{3}{2}$.

The curve cuts the y-axis at $\left(0, \dfrac{3}{2}\right)$.

x-axis: When $y = 0$, $\dfrac{x-3}{x^2+x-2} = 0 \quad \Rightarrow \quad x - 3 = 0$

$\Rightarrow \quad x = 3$

The curve cuts the x-axis at (3, 0).

Vertical Asymptotes: $x^2 + x - 2 = 0 \quad \Rightarrow \quad (x+2)(x-1) = 0$

$\Rightarrow \quad x = -2$ or $x = 1$

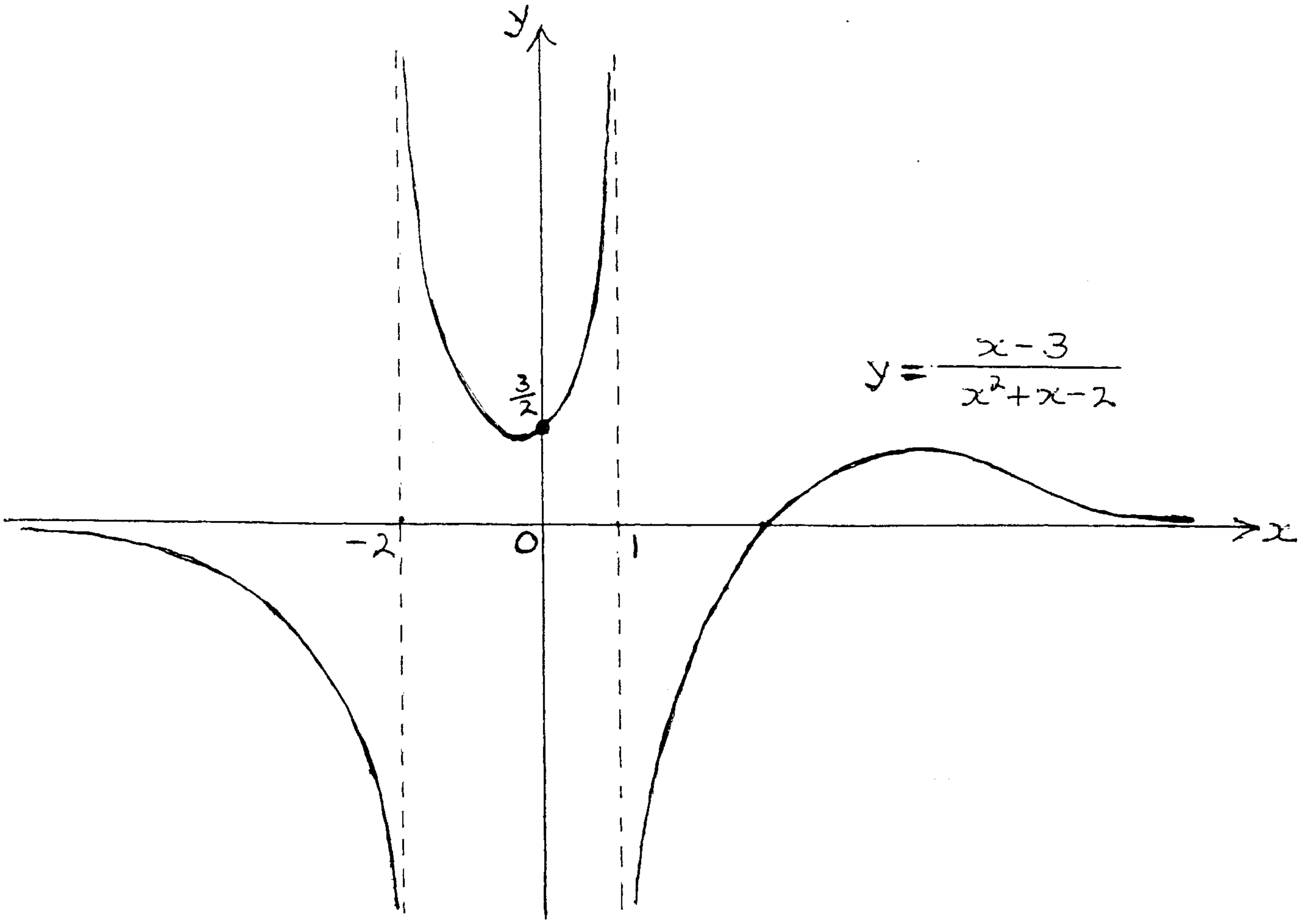
x	$-2{\cdot}1$	-2	$-1{\cdot}9$	$0{\cdot}9$	1	$1{\cdot}1$
y	$-\infty$		$+\infty$	$+\infty$		$-\infty$

Non-Vertical Asymptote:

$$y = \frac{x-3}{x^2+x-2}$$

As $x \to \pm\infty$, $y \to 0$ (since the degree of the denominator is higher than the degree of the numerator).

This means that $y = 0$ is a non-vertical asymptote.



YOU CAN NOW ATTEMPT THE WORKSHEET "SKETCHING RATIONAL FUNCTIONS 1".

Before investigating the non-vertical asymptote of an **improper rational function**, algebraic long division must be used.

Worked Example 3

Sketch the graph of $y = \frac{x+4}{x+2}$.
[You need not find the coordinates of any stationary points.]

Solution

y-axis: When $x = 0$, $y = \frac{4}{2} = 2$.
The curve cuts the y-axis at (0, 2).

x-axis: When $y = 0$, $\frac{x+4}{x+2} = 0 \Rightarrow x + 4 = 0$
$\Rightarrow x = -4$

The curve cuts the x-axis at (–4, 0).

Vertical Asymptotes: $x + 2 = 0 \Rightarrow x = -2$

x	$-2{\cdot}1$	-2	$-1{\cdot}9$
y	$-\infty$		$+\infty$

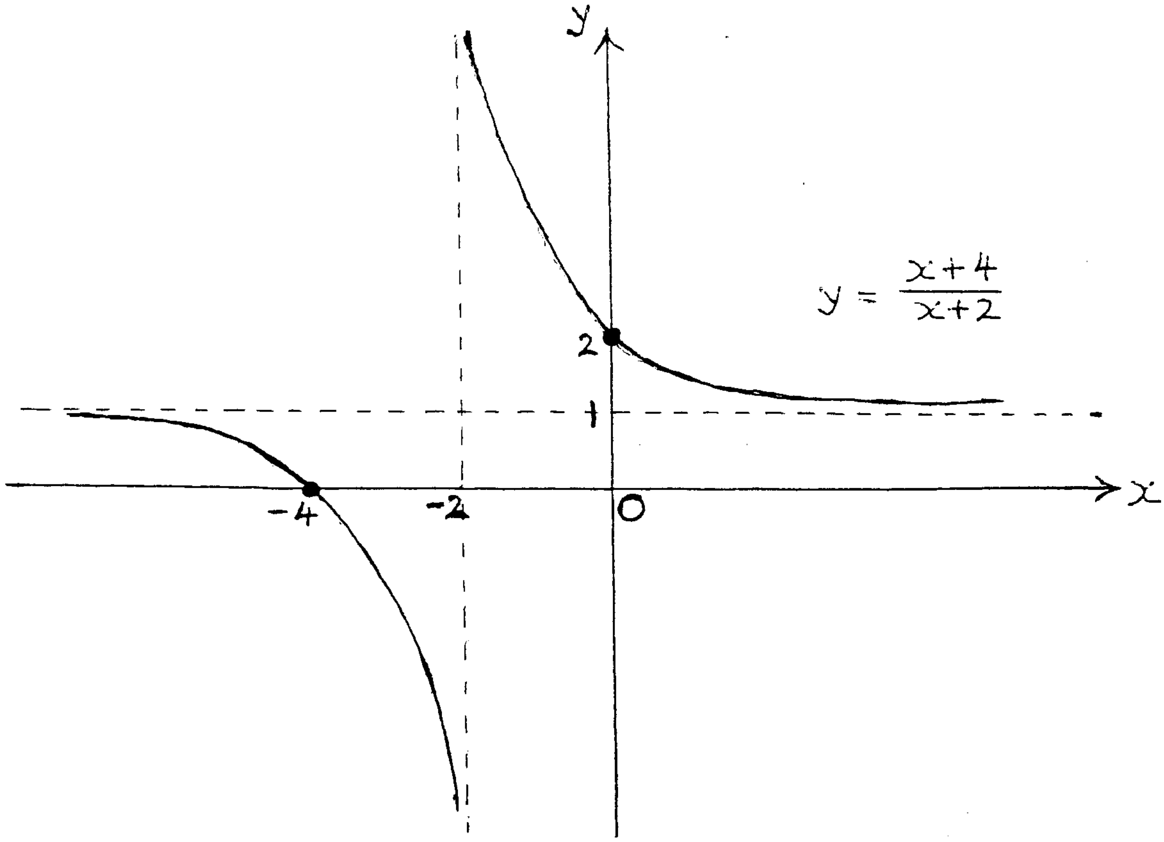
Non-Vertical Asymptote:

$$y = \frac{x+4}{x+2}$$

$$\begin{array}{r|l} & \ \ \ 1 \\ \hline x+2 & x+4 \\ & x+2 \\ \hline & \ \ \ 2 \end{array}$$

$$y = 1 + \frac{2}{x+2}$$

As $x \to \pm\infty$, $y \to 1$.
This means that $y = 1$ is a non-vertical asymptote.



y
x
$y = \frac{x+4}{x+2}$
2
1
-4
-2
0

Worked Example 4

Sketch the graph of $y = \dfrac{x^2 - 4}{x - 1}$.

[You need not find the coordinates of any stationary points.]

Solution

y-axis: When $x = 0$, $y = \dfrac{-4}{-1} = 4$.

The curve cuts the y-axis at (0, 4).

x-axis: When $y = 0$, $\dfrac{x^2 - 4}{x - 1} = 0 \Rightarrow x^2 - 4 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

The curve cuts the x-axis at (–2, 0) and (2, 0).

Vertical Asymptotes: $x - 1 = 0 \Rightarrow x = 1$

x	0·9	1	1·1
y	$+\infty$		$-\infty$

Non-Vertical Asymptote:

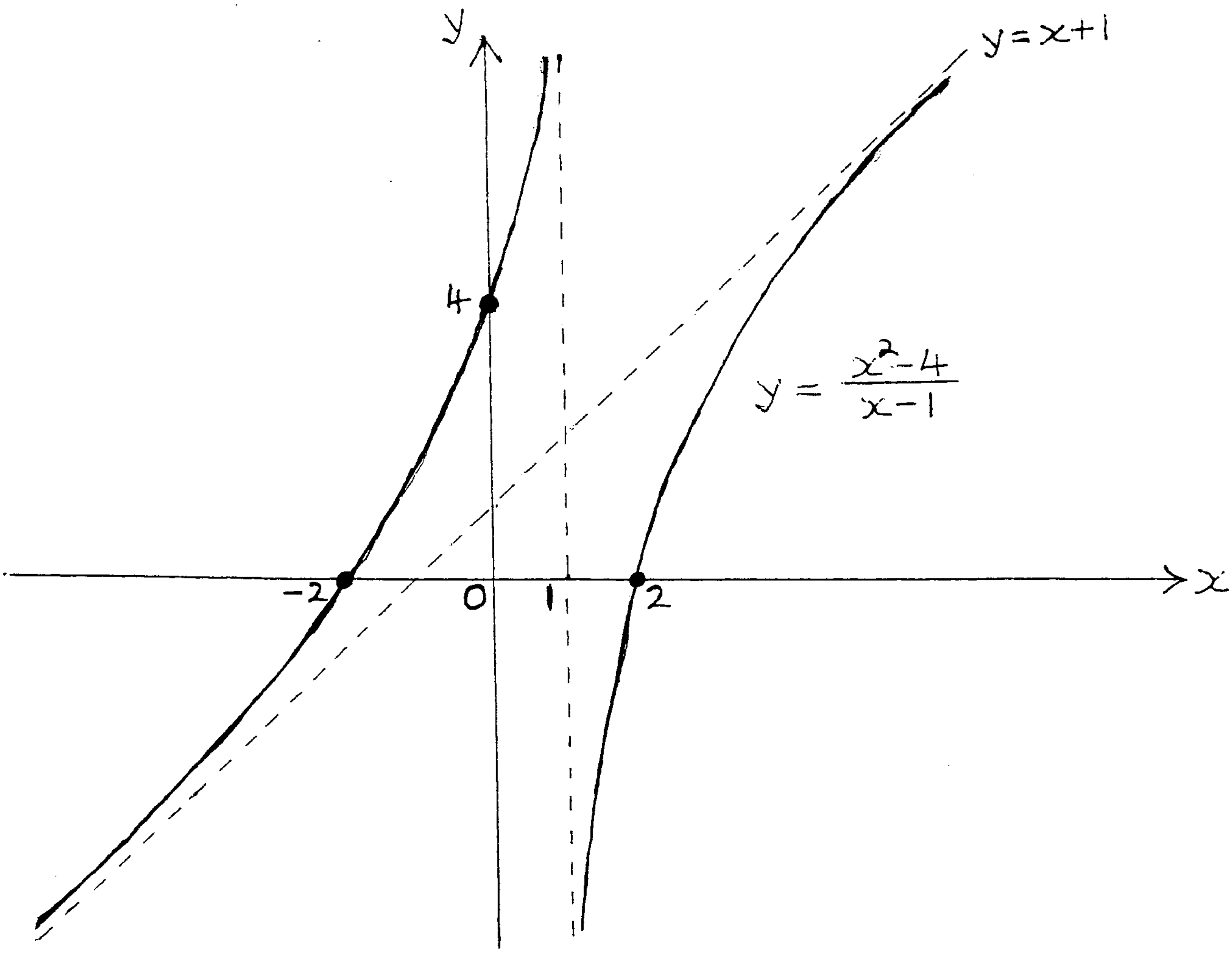
$$y = \frac{x^2 - 4}{x - 1}$$

$$\begin{array}{r|l} & x + 1 \\ \hline x - 1 & x^2 + 0x - 4 \\ & x^2 - x \\ \hline & x - 4 \\ & x - 1 \\ \hline & -3 \end{array}$$

$$y = x + 1 - \frac{3}{x - 1}$$

As $x \to \pm\infty$, $y \to x + 1$.

This means that $y = x + 1$ is a non-vertical asymptote.



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"SKETCHING RATIONAL FUNCTIONS 2".

Worked Example 5

A function f is defined by $f(x) = \dfrac{2x^2 + x - 1}{x - 1}$.

(a) Find the coordinates of all the points where the graph of $y = f(x)$ crosses the coordinate axes.

(b) Find the equation of each asymptote.

(c) Find the coordinates of each of the stationary points on the graph of $y = f(x)$ and determine their nature.

(d) Sketch the graph of $y = f(x)$.

(e) State the range of values of the constant k such that the equation $f(x) = k$ has no real solutions for x.

Solution

(a) *y-axis*: When $x = 0$, $y = \dfrac{-1}{-1} = 1$.

The curve crosses the y-axis at (0, 1).

x-axis: When $y = 0$, $\dfrac{2x^2 + x - 1}{x - 1} = 0 \quad \Rightarrow \quad 2x^2 + x - 1 = 0$

$\Rightarrow \quad (2x - 1)(x + 1) = 0$

$\Rightarrow \quad x = \dfrac{1}{2}$ or $x = -1$

The curve crosses the x-axis at (–1, 0) and $\left(\dfrac{1}{2}, 0\right)$.

(b) *Vertical Asymptotes*: $x - 1 = 0 \quad \Rightarrow \quad x = 1$

x	0·9	1	1·1
y	$-\infty$		$+\infty$

Non-Vertical Asymptote:

$$y = \frac{2x^2 + x - 1}{x - 1}$$

$$\begin{array}{r|l} & \quad\;\; 2x + 3 \\ \hline x - 1 & 2x^2 + \;\; x - 1 \\ & 2x^2 - 2x \\ \hline & \qquad\; 3x - 1 \\ & \qquad\; 3x - 3 \\ \hline & \qquad\qquad\; 2 \end{array}$$

$$y = 2x + 3 + \frac{2}{x - 1}$$

As $x \to \pm\infty$, $y \to 2x + 3$.
This means that $y = 2x + 3$ is a non-vertical asymptote.

(c) There are two methods of finding the coordinates and nature of the stationary points.

Method 1:

The form $y = \frac{2x^2 + x - 1}{x - 1}$ can be differentiated using the quotient rule.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x-1)\cdot(4x+1) - (2x^2 + x - 1)\cdot 1}{(x-1)^2} \\ &= \frac{4x^2 - 3x - 1 - 2x^2 - x + 1}{(x-1)^2} \\ &= \frac{2x^2 - 4x}{(x-1)^2} \\ &= \frac{2x(x-2)}{(x-1)^2} \end{aligned}$$

At a stationary point, $\frac{dy}{dx} = 0 \Rightarrow \quad \frac{2x(x-2)}{(x-1)^2} = 0$

$\Rightarrow \quad 2x(x-2) = 0$

$\Rightarrow \quad x = 0$ or $x = 2$

When $x = 0$, $y = 1 \to (0,1)$.
When $x = 2$, $y = 9 \to (2, 9)$.

The nature of each stationary point can be determined using a nature table.

x	$-0{\cdot}1$	0	$0{\cdot}1$	$1{\cdot}9$	2	$2{\cdot}1$
$\frac{dy}{dx}$	+	0	−	−	0	+
	↗	→	↘	↘	→	↗

(0, 1) is a maximum turning point and (2, 9) is a minimum turning point.

Method 2:

The form $y = 2x + 3 + \dfrac{2}{x-1}$ can be differentiated using the chain rule.

$$y = 2x + 3 + 2(x-1)^{-1} \quad \Rightarrow \quad \frac{dy}{dx} = 2 - 2(x-1)^{-2} \cdot 1$$
$$= 2 - \frac{2}{(x-1)^2}$$

At a stationary point, $\dfrac{dy}{dx} = 0 \Rightarrow \quad 2 - \dfrac{2}{(x-1)^2} = 0 \quad [\times (x-1)^2]$

$$\Rightarrow \quad 2(x-1)^2 - 2 = 0$$
$$\Rightarrow \quad 2(x-1)^2 = 2$$
$$\Rightarrow \quad (x-1)^2 = 1$$
$$\Rightarrow \quad x-1 = 1 \text{ or } x-1 = -1$$
$$\Rightarrow \quad x = 2 \text{ or } x = 0$$

When $x = 0$, $y = 1 \rightarrow (0,1)$.
When $x = 2$, $y = 9 \rightarrow (2, 9)$.

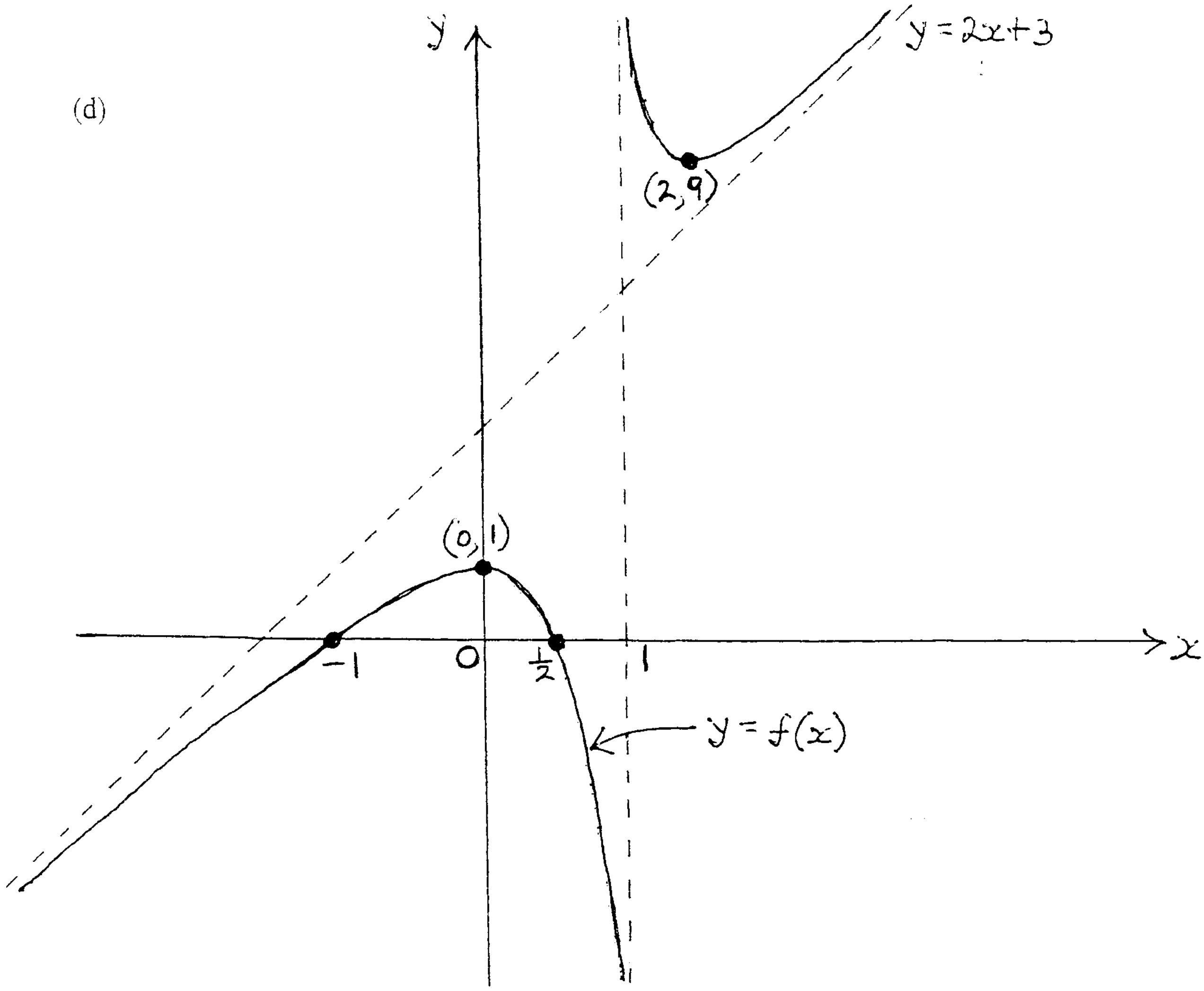
The nature of each stationary point can be found using the second derivative.

$$\frac{dy}{dx} = 2 - 2(x-1)^{-2} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 4(x-1)^{-3} \cdot 1 = \frac{4}{(x-1)^3}$$

When $x = 0$: $\dfrac{d^2y}{dx^2} = \dfrac{4}{(-1)^3} = -4 < 0 \quad \Rightarrow \quad$ (0, 1) is a maximum t.p.

When $x = 2$: $\dfrac{d^2y}{dx^2} = \dfrac{4}{1^3} = 4 > 0 \quad \Rightarrow \quad$ (2, 9) is a minimum t.p.

(d)



(e) The graph shows that the equation $f(x) = k$ has no real solutions when k lies in the interval $1 < k < 9$.

YOU CAN NOW ATTEMPT THE WORKSHEET "SKETCHING RATIONAL FUNCTIONS 3".

THE GRAPH OF $y = |f(x)|$

Recall that $|x|$ denotes the magnitude of a real number x and is the positive numerical value of x, regardless of whether x itself is positive or negative.

$|2| = 2$, $|-3| = 3$, and so on.

Clearly, $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

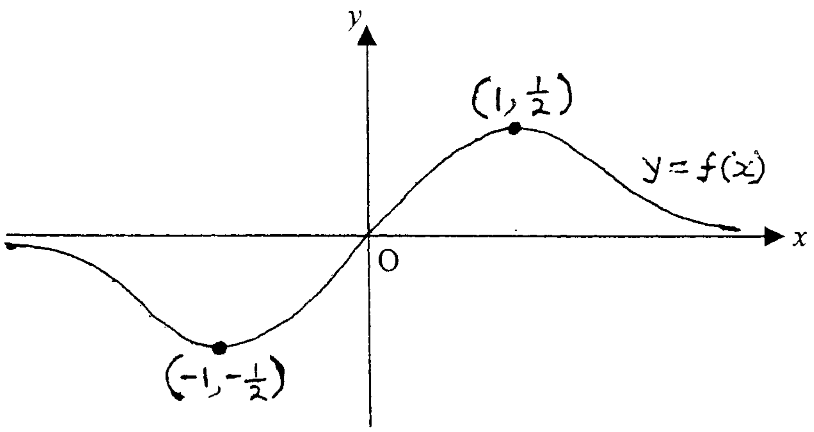
Given a function $f(x)$, $|f(x)|$ is always non-negative and therefore the graph of $y = |f(x)|$ will be lie entirely above or on the x-axis.

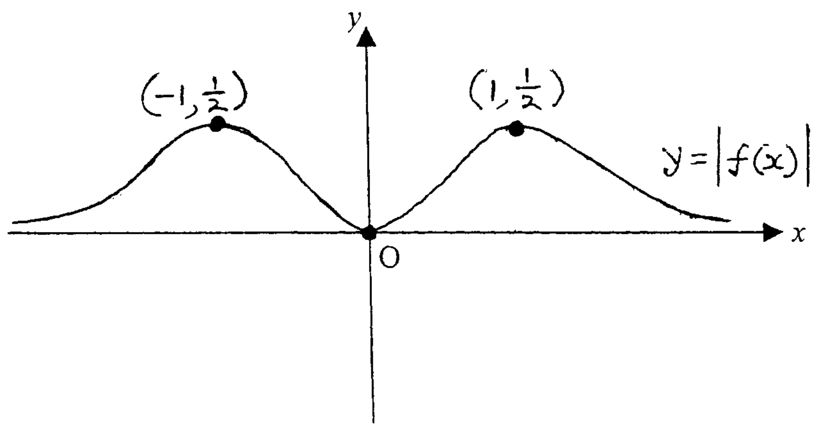
The graph of $y = |f(x)|$ is easily obtained from the graph of $y = f(x)$ as follows:

(1) The parts of the graph of $y = f(x)$ which lie above or on the x-axis will remain unchanged on the graph of $y = |f(x)|$.

(2) The parts of the graph of $y = f(x)$ which lie below the x-axis will must be reflected in the x-axis to lie above the x-axis on the graph of $y = |f(x)|$.

The graphs below illustrate how the graph of $y = |f(x)|$ is obtained from the graph of $y = f(x)$ for a particular function $f(x)$.

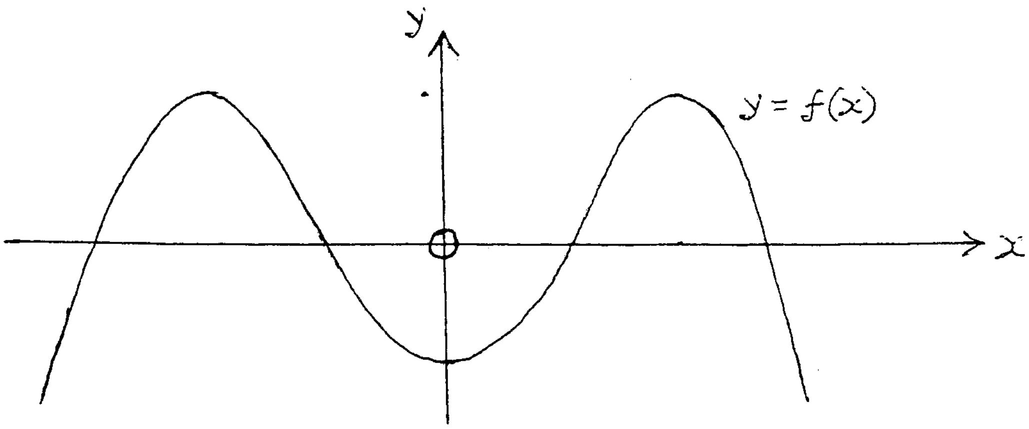




ODD AND EVEN FUNCTIONS

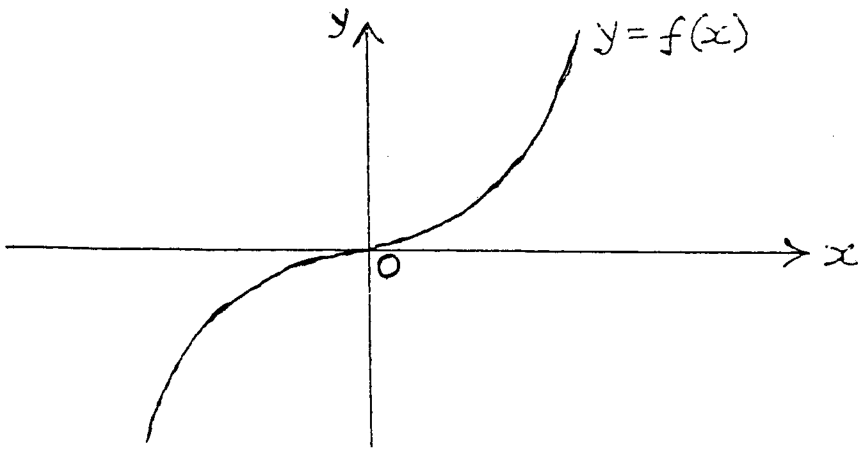
Given a function f, if $f(-x) = f(x)$ for all values of x, f is said to be an **even function**. The graph of an even function will always be symmetrical about the y-axis, since $f(-1) = f(1)$, $f(-2) = f(2)$, etc.

The graph of an even function is shown below.



If $f(-x) = -f(x)$ for all values of x, f is said to be an **odd function**. The graph of an odd function will always have half-turn symmetry about the origin, since $f(-1) = -f(1)$, $f(-2) = -f(2)$, etc.

The graph of an odd function is shown below.



To determine whether a given function f is odd, even or neither, find an expression for $f(-x)$ and compare this expression to $f(x)$. If $f(-x) = f(x)$, then the function f is even; if $f(-x) = -f(x)$, then the function f is odd; otherwise, the function f is neither odd nor even.

It is useful to know the following trigonometric identities for negative angles:

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

You can easily verify using a calculator that, for example, $\sin(-30°) = -\sin 30°$, whereas $\cos(-30°) = \cos 30°$.

Worked Example 1

Prove that the function $f(x) = x^4 - 2x^2 + 3$ is an even function.

Solution

$$\begin{aligned} f(-x) &= (-x)^4 - 2(-x)^2 + 3 \\ &= x^4 - 2x^2 + 3 \quad [\text{since } (-x)^4 = x^4 \text{ and } (-x)^2 = x^2] \\ &= f(x) \end{aligned}$$

Hence $f(-x) = f(x)$ for all values of x and f is an even function.

Worked Example 2

Prove that the function $f(x) = x^3 - 2x$ is an odd function.

Solution

$$\begin{aligned} f(-x) &= (-x)^3 - 2(-x) \\ &= -x^3 + 2x \quad [\text{since } (-x)^3 = -x^3] \\ &= -(x^3 - 2x) \\ &= -f(x) \end{aligned}$$

Hence $f(-x) = -f(x)$ for all values of x and f is an odd function.

Worked Example 3

Investigate whether the function $f(x) = x^3 \sin x$ is odd, even or neither.

Solution

$$\begin{aligned} f(-x) &= (-x)^3 \sin(-x) \\ &= -x^3 \cdot (-\sin x) \qquad [\text{since } (-x)^3 = -x^3 \text{ and } \sin(-x) = -\sin x\,] \\ &= x^3 \sin x \\ &= f(x) \end{aligned}$$

Hence $f(-x) = f(x)$ for all values of x and f is an even function.