FURTHER DIFFERENTIATION

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

$\sin^{-1} x$ is the inverse function of $\sin x$.

For example, we know that $\sin\frac{\pi}{6} = \frac{1}{2}$, so $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Similarly for $\cos^{-1} x$, $\tan^{-1} x$, etc.

The derivative of $\sin^{-1} x$ can be found as follows.

Let $y = \sin^{-1} x$, where $-1 < x < 1$.

$$y = \sin^{-1} x \quad \Rightarrow \quad \sin y = \sin(\sin^{-1} x)$$
$$\Rightarrow \quad \sin y = x \qquad [\text{since } \sin(\sin^{-1} x) = x]$$
$$\Rightarrow \quad x = \sin y$$

Differentiate both sides of this equation **with respect to *y***:

$$\frac{dx}{dy} = \cos y$$

Now $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y}.$$

We must now express $\cos y$ in terms of x, since we require $\frac{dy}{dx}$ to be a function of x.

$$\cos^2 y = 1 - \sin^2 y$$
$$\Rightarrow \quad \cos^2 y = 1 - x^2 \qquad [\text{since } x = \sin y]$$
$$\Rightarrow \quad \cos y = \sqrt{1 - x^2}$$

Hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

[Note that $y = \sin^{-1} x$, where $-1 < x < 1$, so $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and hence $\cos y > 0$.]

A similar method can be used to find the derivatives of $\cos^{-1} x$, $\tan^{-1} x$, etc.

SUMMARY OF DERIVATIVES

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$

Example 1

$y = \sin^{-1}(3x)$

This must be differentiated using the **chain rule**.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 \\ &= \frac{3}{\sqrt{1-9x^2}}\end{aligned}$$

Example 2

$y = \tan^{-1}(x^2)$

This must be differentiated using the **chain rule**.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+(x^2)^2} \cdot 2x \\ &= \frac{2x}{1+x^2}\end{aligned}$$

Example 3

$y = \cos^{-1}(1-2x)$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(1-2x)^2}} \cdot (-2)$$
$$= \frac{2}{\sqrt{1-(1-2x)^2}}$$
$$= \frac{2}{\sqrt{1-(1-4x+4x^2)}}$$
$$= \frac{2}{\sqrt{4x-4x^2}}$$
$$= \frac{2}{\sqrt{4(x-x^2)}}$$
$$= \frac{2}{2\sqrt{x-x^2}} \quad [\text{since } \sqrt{4(x-x^2)} = \sqrt{4}\sqrt{x-x^2} = 2\sqrt{x-x^2}\,]$$
$$= \frac{1}{\sqrt{x-x^2}}$$

Example 4

$y = \tan^{-1}(\sqrt{4x-1}), \quad x > \frac{1}{4}$

This must be differentiated using the **chain rule**.

$$\frac{dy}{dx} = \frac{1}{1+(\sqrt{4x-1})^2} \cdot \frac{d}{dx}\sqrt{4x-1}$$
$$= \frac{1}{1+(4x-1)} \cdot \frac{d}{dx}(4x-1)^{\frac{1}{2}}$$
$$= \frac{1}{4x} \cdot \frac{1}{2}(4x-1)^{-\frac{1}{2}} \cdot 4$$
$$= \frac{1}{2x}(4x-1)^{-\frac{1}{2}}$$
$$= \frac{1}{2x\sqrt{4x-1}}$$

Example 5

$$f(x) = \sin^{-1}\left(\frac{x}{3}\right)$$

This must be differentiated using the **chain rule**.

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \cdot \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{9}}} \cdot \frac{1}{3} \quad \left[\text{since } \frac{d}{dx}\left(\frac{x}{3}\right) = \frac{d}{dx}\left(\frac{1}{3}x\right) = \frac{1}{3}\right]$$

$$= \frac{1}{\sqrt{\frac{9-x^2}{9}}} \cdot \frac{1}{3} \quad \left[\text{since } 1 - \frac{x^2}{9} = \frac{9}{9} - \frac{x^2}{9} = \frac{9-x^2}{9}\right]$$

$$= \frac{1}{\frac{\sqrt{9-x^2}}{3}} \cdot \frac{1}{3} \quad \left[\text{since } \sqrt{\frac{9-x^2}{9}} = \frac{\sqrt{9-x^2}}{\sqrt{9}} = \frac{\sqrt{9-x^2}}{3}\right]$$

$$= \frac{1}{\sqrt{9-x^2}} \quad \left[\text{since } \frac{\sqrt{9-x^2}}{3} \times 3 = \sqrt{9-x^2}\right]$$

Example 6

$$y = x^2 \sin^{-1} x$$

This must be differentiated using the **product rule**.

$$u = x^2 \qquad\qquad v = \sin^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = 2x \qquad\qquad \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^2 \cdot \frac{1}{\sqrt{1-x^2}} + 2x\sin^{-1} x$$

$$= \frac{x^2}{\sqrt{1-x^2}} + 2x\sin^{-1} x$$

Example 7

$y = \sqrt{1-x^2}\cos^{-1} x$

This must be differentiated using the **product rule.**

$$u = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \qquad v = \cos^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) \qquad \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \sqrt{1-x^2} \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) - \frac{x}{\sqrt{1-x^2}} \cdot \cos^{-1} x$$

$$= -1 - \frac{x\cos^{-1} x}{\sqrt{1-x^2}}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS".

IMPLICIT DIFFERENTIATION

Consider the equation $x^2 + y^2 = 1$.

y can be found explicitly in terms of x as follows:

$$x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 - x^2$$
$$\Rightarrow \quad y = \pm\sqrt{1 - x^2}$$

This equation defines y **explicitly** as a function of x.
The equation $x^2 + y^2 = 1$ defines y **implicitly** in terms of x.

When y is defined implicitly in terms of x, it is possible to find an expression for $\frac{dy}{dx}$ without first expressing y explicitly in terms of x. This is useful, since it is often very difficult or impossible to express y explicitly in terms of x.

In general, let y be a function of x.

The derivative of y^2 **with respect to** x is found using the chain rule as follows:

$$\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$$

Similarly, using the chain rule:

$$\frac{d}{dx}(y^3) = 3y^2 \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\sin y = \cos y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}e^y = e^y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}\ln y = \frac{1}{y} \cdot \frac{dy}{dx}$$

Note that xy is actually the product of two functions of x (since y is a function of x) and the derivative of xy **with respect to** x is found using the product rule as follows:

$$\frac{d}{dx}(xy) = x \cdot \frac{dy}{dx} + y \cdot 1$$
$$= x\frac{dy}{dx} + y$$

[Note that the differentiation is performed **with respect to** x and that the derivative of x with respect to x is 1.]

Similarly, using the product rule:

$$\begin{aligned}\frac{d}{dx}(xy^2) &= x \cdot \frac{d}{dx}(y^2) + y^2 \cdot 1 \\ &= x \cdot 2y \cdot \frac{dy}{dx} + y^2 \\ &= 2xy\frac{dy}{dx} + y^2\end{aligned}$$

[Note that $\frac{d}{dx}(y^2)$ is found using the chain rule as before.]

Also using the product rule:

$$\begin{aligned}\frac{d}{dx}(x^2 \cos y) &= x^2 \cdot \frac{d}{dx}\cos y + \cos y \cdot 2x \\ &= x^2 \cdot (-\sin y) \cdot \frac{dy}{dx} + 2x \cos y \\ &= 2x\cos y - x^2 \sin y \frac{dy}{dx}\end{aligned}$$

[Note that $\frac{d}{dx}\cos y$ is found using the chain rule as before.]

Using the product rule again:

$$\begin{aligned}\frac{d}{dx}(xe^y) &= x \cdot \frac{d}{dx}e^y + e^y \cdot 1 \\ &= x \cdot e^y \cdot \frac{dy}{dx} + e^y \\ &= xe^y \frac{dy}{dx} + e^y\end{aligned}$$

And using the product rule once more:

$$\begin{aligned}\frac{d}{dx}(\sin x \ln y) &= \sin x \cdot \frac{d}{dx}\ln y + \ln y \cdot \cos x \\ &= \sin x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \ln y \cos x \\ &= \frac{\sin x}{y}\frac{dy}{dx} + \cos x \ln y\end{aligned}$$

Worked Example 1

The equation $x^2 + y^2 = 4$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$x^2 + y^2 = 4$$

Differentiate both sides of this equation with respect to x:

$$2x + 2y \cdot \frac{dy}{dx} = 0 \qquad [\div 2]$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} = -x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

Worked Example 2

The equation $\ln y = x + y$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$\ln y = x + y$$

Differentiate both sides of this equation with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx} \qquad [\times y \text{ to clear the fraction}]$$

$$\Rightarrow \quad \frac{dy}{dx} = y + y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} - y\frac{dy}{dx} = y$$

$$\Rightarrow \quad \frac{dy}{dx}(1 - y) = y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{1 - y}$$

Worked Example 3

The equation $x^2 + xy = 2$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$x^2 + xy = 2$$

Differentiate both sides of this equation with respect to x:

$$2x + x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x - y}{x}$$

Worked Example 4

The equation $y^2 - xy = x$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$y^2 - xy = x$$

Differentiate both sides of this equation with respect to x:

$$2y \cdot \frac{dy}{dx} - x \cdot \frac{dy}{dx} + y \cdot (-1) = 1$$

$$\Rightarrow \quad 2y\frac{dy}{dx} - x\frac{dy}{dx} - y = 1$$

$$\Rightarrow \quad 2y\frac{dy}{dx} - x\frac{dy}{dx} = 1 + y$$

$$\Rightarrow \quad \frac{dy}{dx}(2y - x) = 1 + y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 + y}{2y - x}$$

Worked Example 5

The equation $x^2y^2 = x^4 + y^4$ defines y implicitly in terms of x.

Find an expression for $\frac{dy}{dx}$ in terms of x and y.

Solution

$$x^2y^2 = x^4 + y^4$$

Differentiate both sides of this equation with respect to x:

$$x^2 \cdot \frac{d}{dx}(y^2) + y^2 \cdot 2x = 4x^3 + 4y^3 \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad x^2 \cdot 2y \cdot \frac{dy}{dx} + 2xy^2 = 4x^3 + 4y^3 \frac{dy}{dx}$$

$$\Rightarrow \quad 2x^2y\frac{dy}{dx} + 2xy^2 = 4x^3 + 4y^3 \frac{dy}{dx} \qquad [\div 2]$$

$$\Rightarrow \quad x^2y\frac{dy}{dx} + xy^2 = 2x^3 + 2y^3 \frac{dy}{dx}$$

$$\Rightarrow \quad x^2y\frac{dy}{dx} - 2y^3 \frac{dy}{dx} = 2x^3 - xy^2$$

$$\Rightarrow \quad \frac{dy}{dx}(x^2y - 2y^3) = 2x^3 - xy^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^3 - xy^2}{x^2y - 2y^3} = \frac{x(2x^2 - y^2)}{y(x^2 - 2y^2)}$$

YOU CAN NOW ATTEMPT QUESTIONS 1 AND 2 OF THE WORKSHEET "IMPLICIT DIFFERENTIATION".

EQUATIONS OF TANGENTS

Worked Example 1

A curve is defined by the implicit equation $x^2 + y^2 + 2x - 4y = 15$.
Find the equation of the tangent at the point (3, 4) on the curve.

Solution

$$x^2 + y^2 + 2x - 4y = 15$$

Differentiate both sides of this equation with respect to x:

$$2x + 2y \cdot \frac{dy}{dx} + 2 - 4\frac{dy}{dx} = 0 \qquad [\div 2]$$

$$\Rightarrow \quad x + y\frac{dy}{dx} + 1 - 2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} - 2\frac{dy}{dx} = -x - 1$$

$$\Rightarrow \quad \frac{dy}{dx}(y - 2) = -x - 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x-1}{y-2}$$

At the point (3, 4): $\frac{dy}{dx} = \frac{-x-1}{y-2} = \frac{-3-1}{4-2} = \frac{-4}{2} = -2 \quad \Rightarrow \quad m_{\text{tangent}} = -2$

Equation of tangent at (3, 4): $y - b = m(x - a)$

$$\Rightarrow \quad y - 4 = -2(x - 3)$$

$$\Rightarrow \quad y - 4 = -2x + 6$$

$$\Rightarrow \quad y = -2x + 10$$

Worked Example 2

A curve is defined by the implicit equation $2x^2 - 3xy - y^2 = 1$.
Find the equation of the tangent at the point (2, 1) on the curve.

Solution

$$2x^2 - 3xy - y^2 = 1$$

Differentiate both sides of this equation with respect to x:

$$4x - 3x \cdot \frac{dy}{dx} + y \cdot (-3) - 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 4x - 3x\frac{dy}{dx} - 3y - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -3x\frac{dy}{dx} - 2y\frac{dy}{dx} = 3y - 4x$$

$$\Rightarrow \quad \frac{dy}{dx}(-3x - 2y) = 3y - 4x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3y - 4x}{-3x - 2y}$$

At the point (2, 1): $\quad \frac{dy}{dx} = \frac{3y - 4x}{-3x - 2y} = \frac{3 \times 1 - 4 \times 2}{-3 \times 2 - 2 \times 1} = \frac{-5}{-8} = \frac{5}{8} \quad \Rightarrow \quad m_{\text{tangent}} = \frac{5}{8}$

Equation of tangent at (2, 1): $\quad y - b = m(x - a)$

$$\Rightarrow \quad y - 1 = \frac{5}{8}(x - 2)$$

$$\Rightarrow \quad 8(y - 1) = 5(x - 2)$$

$$\Rightarrow \quad 8y - 8 = 5x - 10$$

$$\Rightarrow \quad 8y = 5x - 2$$

YOU CAN NOW ATTEMPT QUESTIONS 3 TO 14 OF THE WORKSHEET "IMPLICIT DIFFERENTIATION".

SECOND DERIVATIVES

Worked Example 1

The function $y = f(x)$ is defined implicitly by the equation $x^2 + 2xy = 1$.

Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution

$$x^2 + 2xy = 1$$

Differentiate both sides of this equation with respect to x:

$$2x + 2x \cdot \frac{dy}{dx} + y \cdot 2 = 0 \qquad [\div 2]$$

$$\Rightarrow \quad x + x\frac{dy}{dx} + y = 0 \qquad ...(*)$$

$$\Rightarrow \quad x\frac{dy}{dx} = -x - y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x-y}{x}$$

Note line (*): $x + x\frac{dy}{dx} + y = 0$

To find the second derivative, differentiate line (*) with respect to x (note that the term $x\frac{dy}{dx}$ is the product of two functions of x and must be differentiated using the product rule):

$$1 + x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 1 + \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 1 + x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 1 + x\frac{d^2y}{dx^2} + 2\left(\frac{-x-y}{x}\right) = 0 \qquad [\times x \text{ to clear the fraction}]$$

$$\Rightarrow \quad x + x^2\frac{d^2y}{dx^2} + 2(-x-y) = 0$$

$$\Rightarrow \quad x + x^2\frac{d^2y}{dx^2} - 2x - 2y = 0$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} - x - 2y = 0$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = x + 2y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{x+2y}{x^2}$$

[Alternatively, the second derivative can be found by differentiating the first derivative $\frac{dy}{dx} = \frac{-x-y}{x}$ with respect to x using the quotient rule as follows:

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{x \cdot \frac{d}{dx}(-x-y) - (-x-y) \cdot \frac{d}{dx}(x)}{x^2} \\
&= \frac{x \cdot \left(-1 - \frac{dy}{dx}\right) - (-x-y) \cdot 1}{x^2} \\
&= \frac{-x - x\frac{dy}{dx} + x + y}{x^2} \\
&= \frac{-x\frac{dy}{dx} + y}{x^2} \\
&= \frac{-x\left(\frac{-x-y}{x}\right) + y}{x^2} \\
&= \frac{-(-x-y) + y}{x^2} \\
&= \frac{x + y + y}{x^2} \\
&= \frac{x + 2y}{x^2}
\end{aligned}$$

Generally the previous method is algebraically easier.]

Worked Example 2

The function $y = f(x)$ is defined implicitly by the equation $y^2 - x^2 = 4$.

Find expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

Solution

$$y^2 - x^2 = 4$$

Differentiate both sides of this equation with respect to x:

$$2y \cdot \frac{dy}{dx} - 2x = 0 \qquad [\div 2]$$

$$\Rightarrow \quad y\frac{dy}{dx} - x = 0 \qquad ...(*)$$

$$\Rightarrow \quad y\frac{dy}{dx} = x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}$$

Note line (*): $y\frac{dy}{dx} - x = 0$

To find the second derivative, differentiate line (*) with respect to x (note that the term $y\frac{dy}{dx}$ is the product of two functions of x and must be differentiated using the product rule):

$$y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{dy}{dx} - 1 = 0$$

$$\Rightarrow \quad y \cdot \frac{d^2y}{dx^2} + \frac{x}{y} \cdot \frac{x}{y} - 1 = 0$$

$$\Rightarrow \quad y \cdot \frac{d^2y}{dx^2} + \frac{x^2}{y^2} - 1 = 0 \qquad [\times y^2 \text{ to clear the fraction}]$$

$$\Rightarrow \quad y^3 \frac{d^2y}{dx^2} + x^2 - y^2 = 0$$

$$\Rightarrow \quad y^3 \frac{d^2y}{dx^2} = y^2 - x^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$$

[Alternatively, differentiate the first derivative $\frac{dy}{dx} = \frac{x}{y}$ with respect to x using the quotient rule as follows:

$$\frac{d^2y}{dx^2} = \frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2}$$

$$= \frac{y - x\left(\frac{x}{y}\right)}{y^2}$$

$$= \frac{y - \frac{x^2}{y}}{y^2}$$

$$= \frac{y \times y - \frac{x^2}{y} \times y}{y^2 \times y}$$ [this step can be omitted with practice]

$$= \frac{y^2 - x^2}{y^3}$$]

YOU CAN NOW ATTEMPT QUESTIONS 15 TO 20 OF THE WORKSHEET "IMPLICIT DIFFERENTIATION".

LOGARITHMIC DIFFERENTIATION

Recall the laws of logarithms below:

(1)	$\ln(ab) = \ln a + \ln b$
(2)	$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
(3)	$\ln(a^n) = n \ln a$

When a function involves the product or quotient of powers or roots, **logarithmic differentiation** can be used to find the derivative of the function.

Logarithmic differentiation involves taking the natural logarithm of the function before differentiating. After taking the natural logarithm, the function should be simplified before differentiating.

Worked Example 1

Given $y = 10^x$, use logarithmic differentiation to find $\frac{dy}{dx}$.

Solution

$$y = 10^x \quad \Rightarrow \quad \ln y = \ln(10^x)$$

$$\Rightarrow \quad \ln y = x \ln 10$$

Now differentiate both sides of this equation with respect to x (note that y is a function of x and therefore $\frac{d}{dx}\ln y = \frac{1}{y} \cdot \frac{dy}{dx}$ using the chain rule):

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln 10$$

$$\Rightarrow \quad \frac{dy}{dx} = y \ln 10 = \ln 10 \times 10^x$$

Worked Example 2

Given $y = 2^{3x}$, use logarithmic differentiation to find $\frac{dy}{dx}$.

Solution

$$y = 2^{3x} \quad \Rightarrow \quad \ln y = \ln(2^{3x})$$
$$\Rightarrow \quad \ln y = 3x \ln 2$$

Now differentiate both sides of this equation with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 3 \ln 2$$
$$\Rightarrow \quad \frac{dy}{dx} = y \times 3 \ln 2 = 3 \ln 2 \times 2^{3x}$$

Worked Example 3

Given $y = \frac{x^2 (x+1)^4}{\sqrt{4x+1}}$, use logarithmic differentiation to find $\frac{dy}{dx}$.

Solution

$$y = \frac{x^2 (x+1)^4}{\sqrt{4x+1}} = \frac{x^2 (x+1)^4}{(4x+1)^{\frac{1}{2}}}$$

$$\Rightarrow \quad \ln y = \ln(x^2) + \ln(x+1)^4 - \ln(4x+1)^{\frac{1}{2}}$$

$$\Rightarrow \quad \ln y = 2 \ln x + 4 \ln(x+1) - \frac{1}{2} \ln(4x+1)$$

Now differentiate both sides of this equation with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x} + 4 \cdot \frac{1}{(x+1)} \cdot 1 - \frac{1}{2} \cdot \frac{1}{(4x+1)} \cdot 4$$
$$= \frac{2}{x} + \frac{4}{x+1} - \frac{2}{4x+1}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{2}{x} + \frac{4}{x+1} - \frac{2}{4x+1}\right)$$
$$= \frac{x^2 (x+1)^4}{\sqrt{4x+1}} \left(\frac{2}{x} + \frac{4}{x+1} - \frac{2}{4x+1}\right)$$
$$= \frac{2x^2 (x+1)^4}{\sqrt{4x+1}} \left(\frac{1}{x} + \frac{2}{x+1} - \frac{1}{4x+1}\right)$$

Worked Example 4

Given $y = \dfrac{xe^{x^2}}{\sqrt{\sin x}}$, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

Solution

$$y = \frac{xe^{x^2}}{\sqrt{\sin x}} = \frac{xe^{x^2}}{(\sin x)^{\frac{1}{2}}}$$

$$\Rightarrow \quad \ln y = \ln x + \ln(e^{x^2}) - \ln(\sin x)^{\frac{1}{2}}$$

$$\Rightarrow \quad \ln y = \ln x + x^2 - \frac{1}{2}\ln(\sin x) \qquad [\text{since } \ln(e^{x^2}) = x^2]$$

Now differentiate both sides of the equation with respect to x:

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + 2x - \frac{1}{2}\cdot\frac{1}{\sin x}\cdot\cos x$$

$$= \frac{1}{x} + 2x - \frac{1}{2}\cot x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{1}{x} + 2x - \frac{1}{2}\cot x\right)$$

$$= \frac{xe^{x^2}}{\sqrt{\sin x}}\left(\frac{1}{x} + 2x - \frac{1}{2}\cot x\right)$$

Worked Example 5

Given $y = x^x$, use logarithmic differentiation to find $\dfrac{dy}{dx}$.

Solution

$$y = x^x \quad \Rightarrow \quad \ln y = \ln(x^x)$$

$$\Rightarrow \quad \ln y = x\ln x$$

Now differentiate both sides with respect to x (note that $x\ln x$ is the product of two functions of x and must be differentiated using the product rule):

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln x \cdot 1$$

$$= 1 + \ln x$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \ln x)$$

$$= x^x(1 + \ln x)$$

YOU CAN NOW ATTEMPT THE WORKSHEET "LOGARITHMIC DIFFERENTIATION".

PARAMETRIC DIFFERENTIATION

Consider a point moving in the x, y-plane.

Let (x, y) be the coordinates of the point at time t. Then both x and y are functions of t.

Suppose, for example, that $x = t^2$ and $y = 2t$.

When $t = 3$: $x = 3^2 = 9$ and $y = 2 \times 3 = 6$

Hence the coordinates of the point at time $t = 3$ are (9, 6).

The equations for x and y in terms of t are known as **parametric equations**. t is known as the **parameter**.

An expression for $\frac{dy}{dx}$ in terms of t can be found using the formula below:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$x = t^2 \Rightarrow \frac{dx}{dt} = 2t$ and $y = 2t \Rightarrow \frac{dy}{dt} = 2$

Then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}$.

Worked Example 1

A curve is defined by the parametric equations

$$x = t^2 + \frac{1}{t^2}, \quad y = t^2 - \frac{1}{t^2} \quad (t \neq 0).$$

Find an expression for $\frac{dy}{dx}$ in terms of t.
Give your answer in its simplest form.

Solution

$$x = t^2 + \frac{1}{t^2} = t^2 + t^{-2} \Rightarrow \qquad \frac{dx}{dt} = 2t - 2t^{-3} = 2t - \frac{2}{t^3}$$

$$y = t^2 - \frac{1}{t^2} = t^2 - t^{-2} \Rightarrow \qquad \frac{dy}{dt} = 2t + 2t^{-3} = 2t + \frac{2}{t^3}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + \frac{2}{t^3}}{2t - \frac{2}{t^3}}$$

$$= \frac{2t \times t^3 + \frac{2}{t^3} \times t^3}{2t \times t^3 - \frac{2}{t^3} \times t^3} \qquad \text{[this step can be omitted with practice]}$$

$$= \frac{2t^4 + 2}{2t^4 - 2}$$

$$= \frac{2(t^4 + 1)}{2(t^4 - 1)}$$

$$= \frac{t^4 + 1}{t^4 - 1}$$

Worked Example 2

A curve is defined by the parametric equations

$$x = \theta - \sin\theta, \qquad y = 1 - \cos\theta \quad (0 \le \theta < 2\pi).$$

Find an expression for $\frac{dy}{dx}$ in terms of θ.

Solution

$$x = \theta - \sin\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = 1 - \cos\theta$$

$$y = 1 - \cos\theta \quad \Rightarrow \quad \frac{dy}{d\theta} = -(-\sin\theta) = \sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta}{1 - \cos\theta}$$

Worked Example 3

A curve is defined by the parametric equations

$$x = \frac{t}{1+t}, \qquad y = \frac{1+t}{1-t} \quad (t \ne \pm 1).$$

Find an expression for $\frac{dy}{dx}$ in terms of t.
Give your answer in its simplest form.

Solution

$$x = \frac{t}{1+t} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{(1+t)\cdot 1 - t\cdot 1}{(1+t)^2} = \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2}$$

$$y = \frac{1+t}{1-t} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{(1-t)\cdot 1 - (1+t)\cdot(-1)}{(1-t)^2} = \frac{1-t+1+t}{(1-t)^2} = \frac{2}{(1-t)^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{(1-t)^2}}{\frac{1}{(1+t)^2}} = \frac{2}{(1-t)^2}\cdot\frac{(1+t)^2}{1} = 2\left(\frac{1+t}{1-t}\right)^2$$

YOU CAN NOW ATTEMPT THE WORKSHEET "PARAMETRIC DIFFERENTIATION 1".

EQUATIONS OF TANGENTS

Worked Example 1

A curve is defined by the parametric equations

$$x = t^2 + 1, \quad y = t(t^2 + 1)$$

for all t. Find the equation of the tangent to the curve at the point with parameter $t = 2$.

Solution

$$x = t^2 + 1 \quad \Rightarrow \quad \frac{dx}{dt} = 2t$$

$$y = t(t^2 + 1) = t^3 + t \quad \Rightarrow \quad \frac{dy}{dt} = 3t^2 + 1$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 + 1}{2t}$$

When $t = 2$: $\quad x = t^2 + 1 = 2^2 + 1 = 5$

$$y = t(t^2 + 1) = 2(2^2 + 1) = 2(5) = 10$$

$$\frac{dy}{dx} = \frac{3t^2 + 1}{2t} = \frac{3 \times 2^2 + 1}{2 \times 2} = \frac{13}{4} \quad \Rightarrow \quad m_{\text{tangent}} = \frac{13}{4}$$

Equation of tangent: $\quad y - b = m(x - a)$

$$\Rightarrow \quad y - 10 = \frac{13}{4}(x - 5)$$

$$\Rightarrow \quad 4(y - 10) = 13(x - 5)$$

$$\Rightarrow \quad 4y - 40 = 13x - 65$$

$$\Rightarrow \quad 4y = 13x - 25$$

Worked Example 2

A curve is defined by the parametric equations

$$x = t^2 + t - 1, \qquad y = 2t^2 - t + 2$$

for all t. Show that the point $A(-1, 5)$ lies on the curve and find the equation of the tangent to the curve at the point A.

Solution

At the point $A(-1, 5)$, $x = -1$ and $y = 5$.

When $x = -1$: $t^2 + t - 1 = -1 \Rightarrow t^2 + t = 0$

$\Rightarrow t(t+1) = 0$

$\Rightarrow t = 0$ or $t = -1$

When $y = 5$: $2t^2 - t + 2 = 5 \Rightarrow 2t^2 - t - 3 = 0$

$\Rightarrow (2t-3)(t+1) = 0$

$\Rightarrow t = \frac{3}{2}$ or $t = -1$

$t = -1$ is a common value of the parameter.
$t = -1$ gives $x = -1$ and $y = 5$, so the point $A(-1, 5)$ lies on the curve.

$$x = t^2 + t - 1 \quad \Rightarrow \quad \frac{dx}{dt} = 2t + 1$$

$$y = 2t^2 - t + 2 \quad \Rightarrow \quad \frac{dy}{dt} = 4t - 1$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t-1}{2t+1}$$

When $t = -1$: $\frac{dy}{dx} = \frac{4 \times (-1) - 1}{2 \times (-1) + 1} = \frac{-5}{-1} = 5 \quad \Rightarrow \quad m_{\text{tangent}} = 5$

Equation of tangent at $A(-1, 5)$: $y - b = m(x - a)$

$\Rightarrow y - 5 = 5(x + 1)$

$\Rightarrow y - 5 = 5x + 5$

$\Rightarrow y = 5x + 10$

YOU CAN NOW ATTEMPT QUESTIONS 1 TO 8 OF THE WORKSHEET "PARAMETRIC DIFFERENTIATION 2".

SECOND DERIVATIVES

Consider the curve defined by the parametric equations

$$x = t^2, \quad y = 2t.$$

$$x = t^2 \;\Rightarrow\; \frac{dx}{dt} = 2t$$

$$y = 2t \;\Rightarrow\; \frac{dy}{dt} = 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{2t} = \frac{1}{t}$$

Suppose we wish to find the second derivative $\frac{d^2y}{dx^2}$.

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{t}\right)$.

We can only differentiate $\frac{1}{t}$ with respect to t.

Note that $\frac{d}{dx}\left(\frac{1}{t}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{dt}{dx}$ by the chain rule.

Hence $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{t}\right) \cdot \frac{dt}{dx}$.

$\frac{d}{dt}\left(\frac{1}{t}\right) = \frac{d}{dt}(t^{-1}) = -t^{-2} = -\frac{1}{t^2}$ and $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{2t}$

So $\frac{d^2y}{dx^2} = -\frac{1}{t^2} \cdot \frac{1}{2t} = -\frac{1}{2t^3}$.

In general, let x and y be defined in terms of a parameter t, so that $x = x(t)$ and $y = y(t)$.

Then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ as before.

Now
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}, \quad \text{where } \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}.$$

This gives the general formula below:

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

Worked Example

A curve is defined by the parametric equations

$$x = \frac{2t}{1-t^2}, \quad y = \frac{1+t^2}{1-t^2} \quad (t \neq \pm 1).$$

Show that $\dfrac{dy}{dx} = \dfrac{2t}{1+t^2}$ and $\dfrac{d^2y}{dx^2} = \left(\dfrac{1-t^2}{1+t^2}\right)^3$.

Solution

$$x = \frac{2t}{1-t^2} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{(1-t^2)\cdot 2 - 2t \cdot (-2t)}{(1-t^2)^2}$$
$$= \frac{2 - 2t^2 + 4t^2}{(1-t^2)^2}$$
$$= \frac{2 + 2t^2}{(1-t^2)^2}$$
$$= \frac{2(1+t^2)}{(1-t^2)^2}$$

$$y = \frac{1+t^2}{1-t^2} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{(1-t^2)\cdot 2t - (1+t^2)\cdot(-2t)}{(1-t^2)^2}$$

$$= \frac{2t - 2t^3 + 2t + 2t^3}{(1-t^2)^2}$$

$$= \frac{4t}{(1-t^2)^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{4t}{(1-t^2)^2}}{\frac{2(1+t^2)}{(1-t^2)^2}} = \frac{4t}{(1-t^2)^2} \cdot \frac{(1-t^2)^2}{2(1+t^2)} = \frac{2t}{1+t^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{2t}{1+t^2}\right) = \frac{(1+t^2)\cdot 2 - 2t\cdot 2t}{(1+t^2)^2}$$

$$= \frac{2 + 2t^2 - 4t^2}{(1+t^2)^2}$$

$$= \frac{2 - 2t^2}{(1+t^2)^2}$$

$$= \frac{2(1-t^2)}{(1+t^2)^2}$$

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{(1-t^2)^2}{2(1+t^2)}$$

Hence $\frac{d^2y}{dx^2} = \frac{2(1-t^2)}{(1+t^2)^2} \cdot \frac{(1-t^2)^2}{2(1+t^2)}$

$$= \frac{(1-t^2)^3}{(1+t^2)^3}$$

$$= \left(\frac{1-t^2}{1+t^2}\right)^3.$$

YOU CAN NOW ATTEMPT QUESTIONS 9 TO 11 OF THE WORKSHEET "PARAMETRIC DIFFERENTIATION 2".

STATIONARY POINTS

Recall that the value of the second derivative can be used to determine the nature of a stationary point on a curve.

At a stationary point:

(1) If $\frac{d^2y}{dx^2} > 0$, the stationary point is a **minimum turning point.**

(2) If $\frac{d^2y}{dx^2} < 0$, the stationary point is a **maximum turning point.**

(3) If $\frac{d^2y}{dx^2} = 0$, the second derivative gives **no information** about the nature of the stationary point.
The nature of the stationary point must be determined by another method.

Worked Example

A curve is defined by the parametric equations

$$x = t, \qquad y = t^3 - 3t$$

for all t. Find the coordinates and nature of the stationary points on the curve.

Solution

$$x = t \quad \Rightarrow \quad \frac{dx}{dt} = 1$$

$$y = t^3 - 3t \quad \Rightarrow \quad \frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{1} = 3t^2 - 3$$

At a stationary point, $\frac{dy}{dx} = 0 \Rightarrow \quad 3t^2 - 3 = 0$

$$\Rightarrow \quad 3t^2 = 3$$
$$\Rightarrow \quad t^2 = 1$$
$$\Rightarrow \quad t = \pm 1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(3t^2 - 3) = 6t$$

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{1} = 1$$

Hence $\frac{d^2y}{dx^2} = 6t \cdot 1 = 6t$.

(1) When $t = 1$: $x = t = 1$

$y = t^3 - 3t = 1^3 - 3 \times 1 = -2$

$\frac{d^2y}{dx^2} = 6t = 6 \times 1 = 6 > 0$

Hence $(1, -2)$ is a minimum turning point.

(2) When $t = -1$: $x = t = -1$

$y = t^3 - 3t = (-1)^3 - 3 \times (-1) = 2$

$\frac{d^2y}{dx^2} = 6t = 6 \times (-1) = -6 < 0$

Hence $(-1, 2)$ is a maximum turning point.