FURTHER INTEGRATION

INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Recall that: $\dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)$

$\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2} \quad (x \in \mathbf{R})$

These give the standard integrals below:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

These integrals can be extended as follows:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

You can verify these integrals by differentiation. Use the chain rule to verify that $\dfrac{d}{dx}\sin^{-1}\left(\dfrac{x}{a}\right) = \dfrac{1}{\sqrt{a^2-x^2}}$, where a is a non-zero constant, and that $\dfrac{d}{dx}\left\{\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)\right\} = \dfrac{1}{a^2+x^2}$.

Example 1

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{3^2-x^2}}\,dx$$
$$= \sin^{-1}\left(\frac{x}{3}\right) + C$$

Example 2

$$\int \frac{1}{16+x^2}\,dx = \int \frac{1}{4^2+x^2}\,dx$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$$

Example 3

$$\int \frac{1}{\sqrt{5-x^2}}\,dx = \int \frac{1}{\sqrt{(\sqrt{5})^2-x^2}}\,dx$$
$$= \sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

Example 4

$$\int \frac{4}{x^2+36}\,dx = \int \frac{4}{6^2+x^2}\,dx$$
$$= 4\cdot\frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right) + C$$
$$= \frac{2}{3}\tan^{-1}\left(\frac{x}{6}\right) + C$$

Worked Example 5

Find $\int \frac{1}{\sqrt{25-9x^2}}dx$.

Solution

Method 1 (rewrite the integral in the form $\int \frac{1}{\sqrt{a^2-x^2}}dx$*):*

$$\begin{aligned}\int \frac{1}{\sqrt{25-9x^2}}dx &= \int \frac{1}{\sqrt{9\left(\frac{25}{9}-x^2\right)}}dx \\ &= \int \frac{1}{3\sqrt{\frac{25}{9}-x^2}}dx \\ &= \int \frac{1}{3\sqrt{\left(\frac{5}{3}\right)^2-x^2}}dx \\ &= \frac{1}{3}\sin^{-1}\left(\frac{x}{5/3}\right)+C \qquad \left[\text{using } \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+C\right] \\ &= \frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right)+C\end{aligned}$$

Method 2:

$$\begin{aligned}\int \frac{1}{\sqrt{25-9x^2}}dx &= \int \frac{1}{\sqrt{5^2-(3x)^2}}dx \\ &= \frac{1}{3}\sin^{-1}\left(\frac{3x}{5}\right)+C\end{aligned}$$

Note that since x is replaced by $3x$ in the standard integral, the factor $\frac{1}{3}$ must be included to compensate.

In general, $\int \frac{1}{\sqrt{a^2-(bx+c)^2}}dx = \frac{1}{b}\sin^{-1}\left(\frac{bx+c}{a}\right)+C$.

Worked Example 6

Find $\int \frac{1}{9+4x^2}\,dx$.

Solution

Method 1 (rewrite the integral in the form $\int \frac{1}{a^2+x^2}\,dx$*):*

$$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{4\left(\frac{9}{4}+x^2\right)}dx$$

$$= \int \frac{1}{4\left\{\left(\frac{3}{2}\right)^2+x^2\right\}}dx$$

$$= \frac{1}{4}\cdot\frac{1}{3/2}\cdot\tan^{-1}\left(\frac{x}{3/2}\right)+C \quad \left[\text{using } \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$$

Method 2:

$$\int \frac{1}{9+4x^2}\,dx = \int \frac{1}{3^2+(2x)^2}\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{2x}{3}\right)+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$$

Note that since x is replaced by $2x$ in the standard integral, the factor $\frac{1}{2}$ must be included to compensate.

In general, $\int \frac{1}{a^2+(bx+c)^2}\,dx = \frac{1}{b}\cdot\frac{1}{a}\tan^{-1}\left(\frac{bx+c}{a}\right)+C$.

Example 7

$$\int_{\frac{3}{2}}^{3} \frac{1}{\sqrt{9-x^2}}\,dx = \int_{\frac{3}{2}}^{3} \frac{1}{\sqrt{3^2-x^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{\frac{3}{2}}^{3}$$

$$= \sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{6}$$

$$= \frac{3\pi}{6} - \frac{\pi}{6}$$

$$= \frac{2\pi}{6}$$

$$= \frac{\pi}{3}$$

Example 8

$$\int_{-2}^{2} \frac{1}{4+x^2}\,dx = \int_{-2}^{2} \frac{1}{2^2+x^2}\,dx$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= \frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\tan^{-1}(-1)$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot\left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{8} + \frac{\pi}{8}$$

$$= \frac{2\pi}{8}$$

$$= \frac{\pi}{4}$$

Worked Example 9

Use the substitution $u = x - 3$ to find the indefinite integral $\int \frac{1}{x^2 - 6x + 13}\, dx$.

Solution

$$I = \int \frac{1}{x^2 - 6x + 13}\, dx$$

$$u = x - 3 \quad \Rightarrow \quad \frac{du}{dx} = 1$$

$$\Rightarrow \quad du = dx$$

Now $x^2 - 6x + 13$ must be expressed in terms of u as follows:

$$u = x - 3 \quad \Rightarrow \quad x - 3 = u$$

$$\Rightarrow \quad x = u + 3$$

$$\begin{aligned} x^2 - 6x + 13 &= (u + 3)^2 - 6(u + 3) + 13 \\ &= u^2 + 6u + 9 - 6u - 18 + 13 \\ &= u^2 + 4 \end{aligned}$$

$$\begin{aligned} I &= \int \frac{1}{x^2 - 6x + 13}\, dx \\ &= \int \frac{1}{u^2 + 4}\, du \\ &= \int \frac{1}{2^2 + u^2}\, du \\ &= \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) + C \\ &= \frac{1}{2} \tan^{-1}\left(\frac{x - 3}{2}\right) + C \end{aligned}$$

Worked Example 10

Use the substitution $u = 2\sin x$ to evaluate the definite integral $\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x}dx$.

Solution

$$I = \int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x}dx$$

$$\begin{aligned} u = 2\sin x \quad &\Rightarrow \quad \frac{du}{dx} = 2\cos x \\ &\Rightarrow \quad du = 2\cos x dx \\ &\Rightarrow \quad \frac{1}{2}du = \cos x dx \end{aligned}$$

$$\begin{aligned} 1+4\sin^2 x &= 1 + (2\sin x)^2 \\ &= 1 + u^2 \end{aligned}$$

When $x = 0$: $\quad u = 2\sin 0 = 2\times 0 = 0$

When $x = \frac{\pi}{6}$: $\quad u = 2\sin\frac{\pi}{6} = 2\times\frac{1}{2} = 1$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{6}} \frac{\cos x}{1+4\sin^2 x}dx \\ &= \int_0^{\frac{\pi}{6}} \frac{1}{1+4\sin^2 x}\cdot\cos x dx \\ &= \int_0^{1} \frac{1}{1+u^2}\cdot\frac{1}{2}du \\ &= \left[\frac{1}{2}\tan^{-1} u\right]_0^1 \\ &= \frac{1}{2}\tan^{-1} 1 - \frac{1}{2}\tan^{-1} 0 \\ &= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot 0 \\ &= \frac{\pi}{8} \end{aligned}$$

YOU CAN NOW ATTEMPT THE WORKSHEET "INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS".

USE OF PARTIAL FRACTIONS AND ALGEBRAIC LONG DIVISION IN INTEGRATION

It is often useful to express a rational function in terms of **partial fractions** before integration.

Recall the standard integral $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$.

Worked Example 1

(a) Express $\frac{x+7}{2x^2+3x-2}$ in partial fractions.

(b) Hence find $\int \frac{x+7}{2x^2+3x-2}\,dx$.

Solution

(a) Note that $\frac{x+7}{2x^2+3x-2} = \frac{x+7}{(2x-1)(x+2)}$ and the denominator contains distinct linear factors.

$$\frac{x+7}{(2x-1)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(2x-1)}{(2x-1)(x+2)}$$

$\Rightarrow \quad x+7 = A(x+2)+B(2x-1)$

Put $x=-2 \quad \Rightarrow \quad 5 = A(0)+B(-5)$

$\Rightarrow \quad -5B = 5$

$\Rightarrow \quad B = -1$

Put $x = \frac{1}{2} \quad \Rightarrow \quad 7\frac{1}{2} = A\left(2\frac{1}{2}\right) + B(0)$

$\Rightarrow \quad 2\frac{1}{2}A = 7\frac{1}{2}$

$\Rightarrow \quad A = 3$

Hence $\frac{x+7}{2x^2+3x-2} = \frac{3}{2x-1} - \frac{1}{x+2}$.

(b) $$\int \frac{x+7}{2x^2+3x-2}\,dx = \int \left(\frac{3}{2x-1} - \frac{1}{x+2}\right)dx$$
$$= 3 \cdot \frac{1}{2}\ln|2x-1| - \ln|x+2| + C$$
$$= \frac{3}{2}\ln|2x-1| - \ln|x+2| + C$$

Worked Example 2

(a) Express $\dfrac{1}{x^2 - x - 6}$ in partial fractions.

(b) Hence find the **exact** value of $\displaystyle\int_0^1 \frac{1}{x^2 - x - 6}\,dx$, giving your answer in the form $k \ln a$.

Solution

(a) Note that $\dfrac{1}{x^2 - x - 6} = \dfrac{1}{(x-3)(x+2)}$ and the denominator contains distinct linear factors.

$$\frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$
$$= \frac{A(x+2) + B(x-3)}{(x-3)(x+2)}$$

$\Rightarrow \quad 1 = A(x+2) + B(x-3)$

Put $x = -2 \quad \Rightarrow \quad 1 = A(0) + B(-5)$
$\Rightarrow \quad -5B = 1$
$\Rightarrow \quad B = -\dfrac{1}{5}$

Put $x = 3 \quad \Rightarrow \quad 1 = A(5) + B(0)$
$\Rightarrow \quad 5A = 1$
$\Rightarrow \quad A = \dfrac{1}{5}$

Hence $\dfrac{1}{x^2 - x - 6} = \dfrac{\frac{1}{5}}{x-3} - \dfrac{\frac{1}{5}}{x+2}$. $\left[= \dfrac{1}{5(x-3)} - \dfrac{1}{5(x+2)} \right]$

(b) $$\int_0^1 \frac{1}{x^2 - x - 6}\,dx = \int_0^1 \left(\frac{\frac{1}{5}}{x-3} - \frac{\frac{1}{5}}{x+2} \right) dx$$
$$= \left[\frac{1}{5}\ln|x-3| - \frac{1}{5}\ln|x+2| \right]_0^1$$
$$= \left[\frac{1}{5}\ln|-2| - \frac{1}{5}\ln|3| \right] - \left[\frac{1}{5}\ln|-3| - \frac{1}{5}\ln|2| \right]$$
$$= \left[\frac{1}{5}\ln 2 - \frac{1}{5}\ln 3 \right] - \left[\frac{1}{5}\ln 3 - \frac{1}{5}\ln 2 \right]$$
$$= \frac{1}{5}\ln 2 - \frac{1}{5}\ln 3 - \frac{1}{5}\ln 3 + \frac{1}{5}\ln 2$$

$$= \frac{2}{5}\ln 2 - \frac{2}{5}\ln 3$$
$$= \frac{2}{5}(\ln 2 - \ln 3)$$
$$= \frac{2}{5}\ln\left(\frac{2}{3}\right)$$

YOU CAN NOW ATTEMPT QUESTIONS 1 AND 2 OF THE WORKSHEET "INTEGRATION: USE OF PARTIAL FRACTIONS AND ALGEBRAIC LONG DIVISION".

Before integrating an **improper rational function**, algebraic long division should be used to express the function as the sum of a polynomial and a proper rational function.

Worked Example 3

Find $\int \frac{x^2 + 4x + 2}{x + 1} dx$.

Solution

Note that $\frac{x^2 + 4x + 2}{x + 1}$ is an improper rational function and algebraic long division must be used before integration.

$$
\begin{array}{r|l}
 & x + 3 \\
\hline
x + 1 & x^2 + 4x + 2 \\
 & x^2 + x \\
\hline
 & 3x + 2 \\
 & 3x + 3 \\
\hline
 & -1
\end{array}
$$

$$\frac{x^2 + 4x + 2}{x + 1} = x + 3 - \frac{1}{x + 1}$$

$$\int \frac{x^2 + 4x + 2}{x + 1} dx = \int \left(x + 3 - \frac{1}{x + 1} \right) dx$$

$$= \frac{1}{2}x^2 + 3x - \ln|x + 1| + C$$

YOU CAN NOW ATTEMPT QUESTIONS 3 AND 4 OF THE WORKSHEET "INTEGRATION: USE OF PARTIAL FRACTIONS AND ALGEBRAIC LONG DIVISION".

Sometimes a combination of algebraic long division and partial fractions must be used before integrating an improper rational function.

Worked Example 4

(a) Express $\frac{x}{x^2-1}$ in partial fractions.

(b) Hence find $\int \frac{x^3}{x^2-1}\,dx$.

Solution

(a) Note that $\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)}$ and the denominator contains distinct linear factors.

$$\frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$
$$= \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$\Rightarrow \quad x = A(x+1) + B(x-1)$

Put $x=-1$ $\Rightarrow$ $-1 = A(0) + B(-2)$

$\Rightarrow$ $-2B = -1$

$\Rightarrow$ $B = \frac{1}{2}$

Put $x = 1$ $\Rightarrow$ $1 = A(2) + B(0)$

$\Rightarrow$ $2A = 1$

$\Rightarrow$ $A = \frac{1}{2}$

Hence $\frac{x}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}$. $\left[= \frac{1}{2(x-1)} + \frac{1}{2(x+1)} \right]$

(b) Note that $\dfrac{x^3}{x^2-1}$ is an improper rational function and algebraic long division must be used before integration.

$$\begin{array}{r|l} & \quad\quad\quad\quad x \\ x^2-1 & x^3+0x^2+0x+0 \\ & x^3 \quad\quad\quad - x \\ \hline & \quad\quad\quad\quad\quad x \end{array}$$

$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$$

$$= x + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}$$ [inserting the partial fractions for $\dfrac{x}{x^2-1}$ from (a)]

Hence $$\int \frac{x^3}{x^2-1}\,dx = \int\left(x + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\right)dx$$

$$= \frac{1}{2}x^2 + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

YOU CAN NOW ATTEMPT QUESTION 5 OF THE WORKSHEET "INTEGRATION: USE OF PARTIAL FRACTIONS AND ALGEBRAIC LONG DIVISION".

Miscellaneous Example 5

(a) Express $\dfrac{1}{x^3 + x}$ in partial fractions.

(b) Hence obtain a formula for $I(k) = \displaystyle\int_1^k \frac{1}{x^3 + x}\,dx$, where $k > 1$, expressing your formula in the form $\ln\left(\dfrac{a}{b}\right)$, where a and b are functions of k.

Solution

(a) Note that $\dfrac{1}{x^3 + x} = \dfrac{1}{x(x^2 + 1)}$.

We must use the discriminant to verify that the quadratic factor $x^2 + 1$ is irreducible.

For $x^2 + 1$: $a = i$, $b = 0$, $c = 1$ $\Rightarrow$ $b^2 - 4ac = 0^2 - 4 \times 1 \times 1 = -4$

$b^2 - 4ac < 0$, hence $x^2 + 1$ is irreducible.

$$\frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$
$$= \frac{A(x^2 + 1) + (Bx + C)x}{x(x^2 + 1)}$$

$\Rightarrow$ $1 = A(x^2 + 1) + (Bx + C)x$

Put $x = 0$ $\Rightarrow$ $1 = A(1) + 0$
$\Rightarrow$ $A = 1$

Put $x = 1$ $\Rightarrow$ $1 = A(2) + (B + C)(1)$
$\Rightarrow$ $2A + B + C = 1$
$\Rightarrow$ $2 + B + C = 1$
$\Rightarrow$ $B + C = -1$ ...(1)

Put $x = 2$ $\Rightarrow$ $1 = A(5) + (2B + C)(2)$
$\Rightarrow$ $5A + 4B + 2C = 1$
$\Rightarrow$ $5 + 4B + 2C = 1$
$\Rightarrow$ $4B + 2C = -4$ ...(2)

Solving equations (1) and (2) simultaneously gives $B = -1$ and $C = 0$.

Hence $\dfrac{1}{x^3 + x} = \dfrac{1}{x} + \dfrac{-1x + 0}{x^2 + 1}$
$$= \frac{1}{x} - \frac{x}{x^2 + 1}$$

(b) $$I(k)=\int_1^k \frac{1}{x^3+x}\,dx$$
$$=\int_1^k\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$$

We know that $\int \frac{1}{x}dx=\ln|x|+C$.

To find $\int \frac{x}{x^2+1}dx$, use the substitution $u=x^2+1$:

$$u=x^2+1 \quad\Rightarrow\quad \frac{du}{dx}=2x$$
$$\Rightarrow\quad du=2xdx$$
$$\Rightarrow\quad \frac{1}{2}du=xdx$$

$$\int\frac{x}{x^2+1}dx=\int\frac{1}{x^2+1}\cdot xdx$$
$$=\int\frac{1}{u}\cdot\frac{1}{2}du$$
$$=\frac{1}{2}\ln|u|+C$$
$$=\frac{1}{2}\ln|x^2+1|+C$$

[Note that $x^2+1>0$ for all values of x, hence $|x^2+1|=x^2+1$ and the answer can be expressed simply as $\frac{1}{2}\ln(x^2+1)+C$ if required.]

Now $$I(k)=\int_1^k\left(\frac{1}{x}-\frac{x}{x^2+1}\right)dx$$
$$=\left[\ln|x|-\frac{1}{2}\ln|x^2+1|\right]_1^k$$
$$=\left[\ln k-\frac{1}{2}\ln(k^2+1)\right]-\left[\ln 1-\frac{1}{2}\ln 2\right]$$
$$=\ln k-\frac{1}{2}\ln(k^2+1)+\frac{1}{2}\ln 2 \qquad [\text{since } \ln 1=0]$$
$$=\ln k-\ln(k^2+1)^{\frac{1}{2}}+\ln(2^{\frac{1}{2}})$$
$$=\ln k-\ln\sqrt{k^2+1}+\ln\sqrt{2}$$
$$=\ln\left(\frac{\sqrt{2}k}{k^2+1}\right)$$

INTEGRATION BY PARTS

Let u and v be functions of x.

Then $\qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ $\qquad$ [by the product rule]

$\Rightarrow \qquad u\frac{dv}{dx} + v\frac{du}{dx} = \frac{d}{dx}(uv)$

$\Rightarrow \qquad u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

Now integrate both sides with respect to x:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

This formula can be used to integrate the product of two functions of x and is known as **integration by parts**.

In order to use integration by parts, one of the functions must be easy to differentiate and the other function must be easy to integrate.

Worked Example 1

Use integration by parts to find $\int x\cos 2x dx$.

Solution

$I = \int x\cos 2x dx$

$u = x \qquad \frac{dv}{dx} = \cos 2x$

$\Rightarrow \qquad \frac{du}{dx} = 1 \qquad v = \frac{1}{2}\sin 2x$

$$I = uv - \int v\frac{du}{dx}dx$$

$$= \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x dx$$

$$= \frac{1}{2}x\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + C$$

$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

Worked Example 2

Use integration by parts to find $\int x \sin 3x dx$.

Solution

$$I = \int x \sin 3x dx$$

$$u = x \qquad \frac{dv}{dx} = \sin 3x$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad v = -\frac{1}{3}\cos 3x$$

$$\begin{aligned}
I &= uv - \int v \frac{du}{dx} dx \\
&= -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x dx \\
&= -\frac{1}{3}x\cos 3x + \int \frac{1}{3}\cos 3x dx \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{3}\cdot\frac{1}{3}\sin 3x + C \\
&= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C
\end{aligned}$$

Worked Example 3

Use integration by parts to find $\int xe^{4x} dx$.

Solution

$$I = \int xe^{4x} dx$$

$$u = x \qquad \frac{dv}{dx} = e^{4x}$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad v = \frac{1}{4}e^{4x}$$

$$\begin{aligned}
I &= uv - \int v \frac{du}{dx} dx \\
&= \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x} dx \\
&= \frac{1}{4}xe^{4x} - \frac{1}{4}\cdot\frac{1}{4}e^{4x} + C \\
&= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C
\end{aligned}$$

Worked Example 4

Use integration by parts to find $\int(2x+1)e^{-2x}dx$.

Solution

$$I = \int(2x+1)e^{-2x}dx$$

$$u = 2x+1 \qquad \frac{dv}{dx} = e^{-2x}$$

$$\Rightarrow \quad \frac{du}{dx} = 2 \qquad v = -\frac{1}{2}e^{-2x}$$

$$\begin{aligned} I &= uv - \int v\frac{du}{dx}dx \\ &= -\frac{1}{2}(2x+1)e^{-2x} - \int -e^{-2x}dx \\ &= -\frac{1}{2}(2x+1)e^{-2x} + \int e^{-2x}dx \\ &= -\frac{1}{2}(2x+1)e^{-2x} - \frac{1}{2}e^{-2x} + C \end{aligned}$$

Worked Example 5

Use integration by parts to find $\int x^3 \ln x dx$.

Solution

$$I = \int x^3 \ln x dx$$

We must select $\ln x$ as the function to be differentiated, as the standard integral for $\int \ln x dx$ is not known.

$$u = \ln x \qquad \frac{dv}{dx} = x^3$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{4}x^4$$

$$\begin{aligned} I &= uv - \int v\frac{du}{dx}dx \\ &= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4}\cdot\frac{1}{4}x^4 + C \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C \end{aligned}$$

Worked Example 6

Use integration by parts to find $\int \frac{1}{x^4} \ln x dx$.

Solution

$$I = \int \frac{1}{x^4} \ln x dx$$

$$u = \ln x \qquad \frac{dv}{dx} = \frac{1}{x^4} = x^{-4}$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{-3} x^{-3} = -\frac{1}{3x^3}$$

$$\begin{aligned}
I &= uv - \int v \frac{du}{dx} dx \\
&= -\frac{1}{3x^3} \ln x - \int -\frac{1}{3x^4} dx \\
&= -\frac{1}{3x^3} \ln x + \int \frac{1}{3x^4} dx \\
&= -\frac{1}{3x^3} \ln x + \int \frac{1}{3} x^{-4} dx \\
&= -\frac{1}{3x^3} \ln x + \frac{1}{3} \cdot \frac{1}{(-3)} x^{-3} + C \\
&= -\frac{1}{3x^3} \ln x - \frac{1}{9} x^{-3} + C \\
&= -\frac{1}{3x^3} \ln x - \frac{1}{9x^3} + C
\end{aligned}$$

Worked Example 7

Use integration by parts to find $\int x(2x+1)^4\,dx$.

Solution

$$I = \int x(2x+1)^4 dx$$

$$u = x \qquad\qquad \frac{dv}{dx} = (2x+1)^4$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad\qquad v = \frac{1}{2}\cdot\frac{1}{5}(2x+1)^5 = \frac{1}{10}(2x+1)^5$$

$$\begin{aligned}
I &= uv - \int v\frac{du}{dx}\,dx \\
&= \frac{1}{10}x(2x+1)^5 - \int \frac{1}{10}(2x+1)^5\,dx \\
&= \frac{1}{10}x(2x+1)^5 - \frac{1}{10}\cdot\frac{1}{2}\cdot\frac{1}{6}(2x+1)^6 + C \\
&= \frac{1}{10}x(2x+1)^5 - \frac{1}{120}(2x+1)^6 + C
\end{aligned}$$

Worked Example 8

Use integration by parts to find $\int x\sqrt{2x-1}\,dx$.

Solution

$$I = \int x\sqrt{2x-1}\,dx$$

$$u = x \qquad\qquad \frac{dv}{dx} = \sqrt{2x-1} = (2x-1)^{\frac{1}{2}}$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad\qquad v = \frac{1}{2}\cdot\frac{2}{3}(2x-1)^{\frac{3}{2}} = \frac{1}{3}(2x-1)^{\frac{3}{2}}$$

$$\begin{aligned}
I &= uv - \int v\frac{du}{dx}\,dx \\
&= \frac{1}{3}x(2x-1)^{\frac{3}{2}} - \int \frac{1}{3}(2x-1)^{\frac{3}{2}}\,dx \\
&= \frac{1}{3}x(2x-1)^{\frac{3}{2}} - \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{5}(2x-1)^{\frac{5}{2}} + C \\
&= \frac{1}{3}x(2x-1)^{\frac{3}{2}} - \frac{1}{15}(2x-1)^{\frac{5}{2}} + C
\end{aligned}$$

Worked Example 9

Use integration by parts to find $\int \frac{x}{\sqrt{x+4}}dx$.

Solution

$$I = \int \frac{x}{\sqrt{x+4}}dx = \int x \cdot \frac{1}{\sqrt{x+4}}dx$$

$$u = x \qquad \frac{dv}{dx} = \frac{1}{\sqrt{x+4}} = (x+4)^{-\frac{1}{2}}$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad v = 2(x+4)^{\frac{1}{2}} = 2\sqrt{x+4}$$

$$\begin{aligned} I &= uv - \int v\frac{du}{dx}dx \\ &= 2x\sqrt{x+4} - \int 2\sqrt{x+4}dx \\ &= 2x\sqrt{x+4} - \int 2(x+4)^{\frac{1}{2}}dx \\ &= 2x\sqrt{x+4} - 2 \cdot \frac{2}{3}(x+4)^{\frac{3}{2}} + C \\ &= 2x\sqrt{x+4} - \frac{4}{3}(x+4)^{\frac{3}{2}} + C \end{aligned}$$

Worked Example 10

Use integration by parts to find $\int \ln x dx$.

Solution

$$I = \int \ln x dx = \int 1 \cdot \ln x dx$$

We must select $\ln x$ as the function to be differentiated, as the standard integral for $\int \ln x dx$ is not known.

$$u = \ln x \qquad \frac{dv}{dx} = 1$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{x} \qquad v = x$$

$$\begin{aligned} I &= uv - \int v\frac{du}{dx}dx \\ &= x\ln x - \int 1dx \\ &= x\ln x - x + C \end{aligned}$$

Worked Example 11

Use integration by parts to find $\int \sin^{-1} x dx$.

Solution

$$I = \int \sin^{-1} x dx = \int 1 \cdot \sin^{-1} x dx$$

We must select $\sin^{-1} x$ as the function to be differentiated, as the standard integral for $\int \sin^{-1} x dx$ is not known.

$$u = \sin^{-1} x \qquad \frac{dv}{dx} = 1$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad v = x$$

$$I = uv - \int v \frac{du}{dx} dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx \quad ...(*)$$

To find $\int \frac{x}{\sqrt{1-x^2}} dx$, use the substitution $u = 1 - x^2$:

$$u = 1 - x^2 \quad \Rightarrow \quad \frac{du}{dx} = -2xdx$$

$$\Rightarrow \quad du = -2xdx$$

$$\Rightarrow \quad -\frac{1}{2} du = xdx$$

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-x^2}} \cdot xdx$$

$$= \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{2} du\right)$$

$$= \int -\frac{1}{2} u^{-\frac{1}{2}} du$$

$$= -\frac{1}{2} \cdot 2u^{\frac{1}{2}}$$ [there is no need to include a constant of integration here]

$$= -\sqrt{u}$$

$$= -\sqrt{1-x^2}$$

From line (*): $I = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$

$$= x \sin^{-1} x - (-\sqrt{1-x^2}) + C$$ [include the constant of integration now]

$$= x \sin^{-1} x + \sqrt{1-x^2} + C$$

Worked Example 12

Use integration by parts to find $\int \tan^{-1} x dx$.

Solution

$I = \int \tan^{-1} x dx = \int 1 \cdot \tan^{-1} x dx$

We must select $\tan^{-1} x$ as the function to be differentiated, as the standard integral for $\int \tan^{-1} x dx$ is not known.

$$u = \tan^{-1} x \qquad \frac{dv}{dx} = 1$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{1+x^2} \qquad v = x$$

$$I = uv - \int v \frac{du}{dx} dx$$

$$= x \tan^{-1} x - \int \frac{x}{1+x^2} dx \qquad ...(*)$$

To find $\int \frac{x}{1+x^2} dx$, use the substitution $u = 1 + x^2$:

$$u = 1 + x^2 \quad \Rightarrow \quad \frac{du}{dx} = 2xdx$$

$$\Rightarrow \quad du = 2xdx$$

$$\Rightarrow \quad \frac{1}{2} du = xdx$$

$$\int \frac{x}{1+x^2} dx = \int \frac{1}{1+x^2} \cdot xdx$$

$$= \int \frac{1}{u} \cdot \frac{1}{2} du$$

$= \frac{1}{2} \ln|u|$ [there is no need to include a constant of integration here]

$$= \frac{1}{2} \ln|1 + x^2|$$

$= \frac{1}{2} \ln(1 + x^2)$ [since $1 + x^2 > 0$ for all values of x]

From line (*): $I = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$

$= x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$ [include the constant of integration now]

YOU CAN NOW ATTEMPT QUESTIONS 1 TO 8 OF THE WORKSHEET "INTEGRATION BY PARTS".

Worked Example 13

Use integration by parts **twice** to find $\int x^2 \cos 3x dx$.

Solution

$$I = \int x^2 \cos 3x dx$$

$$u = x^2 \qquad \frac{dv}{dx} = \cos 3x$$

$$\Rightarrow \quad \frac{du}{dx} = 2x \qquad v = \frac{1}{3}\sin 3x$$

$$\begin{aligned} I &= uv - \int v\frac{du}{dx}dx \\ &= \frac{1}{3}x^2 \sin 3x - \int \frac{2}{3}x \sin 3x dx \qquad \text{...(*)} \end{aligned}$$

To find $\int \frac{2}{3}x \sin 3x dx$, use integration by parts again:

$$u = \frac{2}{3}x \qquad \frac{dv}{dx} = \sin 3x$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{2}{3} \qquad v = -\frac{1}{3}\cos 3x$$

$$\begin{aligned} \int \frac{2}{3}x \sin 3x dx &= uv - \int v\frac{du}{dx}dx \\ &= -\frac{2}{9}x\cos 3x - \int -\frac{2}{9}\cos 3x dx \\ &= -\frac{2}{9}x\cos 3x + \int \frac{2}{9}\cos 3x dx \\ &= -\frac{2}{9}x\cos 3x + \frac{2}{9}\cdot\frac{1}{3}\sin 3x \qquad \text{[there is no need to include a constant of integration here]} \\ &= -\frac{2}{9}x\cos 3x + \frac{2}{27}\sin 3x \end{aligned}$$

From line (*):

$$\begin{aligned} I &= \frac{1}{3}x^2 \sin 3x - \int \frac{2}{3}x \sin 3x dx \\ &= \frac{1}{3}x^2 \sin 3x - \left(-\frac{2}{9}x\cos 3x + \frac{2}{27}\sin 3x\right) + C \\ &= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + C \end{aligned}$$

YOU CAN NOW ATTEMPT QUESTIONS 9 AND 10 OF THE WORKSHEET "INTEGRATION BY PARTS".

Worked Example 14

Use integration by parts to evaluate the definite integral $\int_0^{\frac{\pi}{2}} x \sin 2x dx$.

Solution

$$I = \int x \sin 2x dx$$

$$u = x \qquad \frac{dv}{dx} = \sin 2x$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad v = -\frac{1}{2}\cos 2x$$

$$\begin{aligned}
I &= uv - \int v \frac{du}{dx} dx \\
&= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x dx \\
&= -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x dx \\
&= -\frac{1}{2}x\cos 2x + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + C \\
&= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C
\end{aligned}$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x\sin 2x dx &= \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} \\
&= \left[-\frac{1}{2}\cdot\frac{\pi}{2}\cdot\cos\pi + \frac{1}{4}\sin\pi\right] - \left[-\frac{1}{2}\cdot 0\cdot\cos 0 + \frac{1}{4}\sin 0\right] \\
&= \left[-\frac{\pi}{4}(-1) + \frac{1}{4}(0)\right] - [0] \\
&= \frac{\pi}{4}
\end{aligned}$$

Worked Example 15

Use integration by parts to evaluate the definite integral $\int_0^1 xe^{-2x}dx$, expressing your answer in terms of e.

Solution

$$I = \int xe^{-2x}dx$$

$$u = x \qquad \frac{dv}{dx} = e^{-2x}$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad v = -\frac{1}{2}e^{-2x}$$

$$\begin{aligned} I &= uv - \int v\frac{du}{dx}dx \\ &= -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}dx \\ &= -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}dx \\ &= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C \end{aligned}$$

$$\begin{aligned} \int_0^1 xe^{-2x}dx &= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1 \\ &= \left[-\frac{1}{2}\cdot 1\cdot e^{-2} - \frac{1}{4}e^{-2}\right] - \left[-\frac{1}{2}\cdot 0\cdot e^{0} - \frac{1}{4}e^{0}\right] \\ &= \left[-\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2}\right] - \left[0 - \frac{1}{4}\right] \\ &= -\frac{3}{4}e^{-2} + \frac{1}{4} \\ &= \frac{1}{4}(1 - 3e^{-2}) \end{aligned}$$

YOU CAN NOW ATTEMPT QUESTIONS 11 AND 12 OF THE WORKSHEET "INTEGRATION BY PARTS".

Worked Example 16

Use integration by parts **twice** to find $\int e^x \cos x dx$.

Solution

$I = \int e^x \cos x dx$

$u = e^x \qquad\qquad \dfrac{dv}{dx} = \cos x$

$\Rightarrow \quad \dfrac{du}{dx} = e^x \qquad\qquad v = \sin x$

$$I = uv - \int v \frac{du}{dx} dx$$
$$= e^x \sin x - \int e^x \sin x dx \qquad ...(*)$$

To find $\int e^x \sin x dx$, use integration by parts again:

$u = e^x \qquad\qquad \dfrac{dv}{dx} = \sin x$

$\Rightarrow \quad \dfrac{du}{dx} = e^x \qquad\qquad v = -\cos x$

$$\int e^x \sin x dx = uv - \int v \frac{du}{dx} dx$$
$$= -e^x \cos x - \int -e^x \cos x dx$$
$$= -e^x \cos x + \int e^x \cos x dx$$
$$= -e^x \cos x + I$$

From line (*):

$$I = e^x \sin x - \int e^x \sin x dx$$
$$= e^x \sin x - (-e^x \cos x + I)$$
$$= e^x \sin x + e^x \cos x - I$$

$$\Rightarrow \quad 2I = e^x (\sin x + \cos x)$$
$$\Rightarrow \quad I = \frac{1}{2} e^x (\sin x + \cos x) + C$$

YOU CAN NOW ATTEMPT QUESTION 13 OF THE WORKSHEET "INTEGRATION BY PARTS".

Miscellaneous Example 17

Define $I_n = \int_0^1 x^n e^{-x} dx$ for $n \geq 1$.

(a) Use integration by parts to evaluate I_1.

(b) Show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.

(c) Using your answers to parts (a) an (b), evaluate I_3.

Solution

(a) $I_1 = \int_0^1 x^1 e^{-x} dx = \int_0^1 xe^{-x} dx$

$$u = x \qquad \frac{dv}{dx} = e^{-x}$$

$$\Rightarrow \quad \frac{du}{dx} = 1 \qquad v = -e^{-x}$$

$$I = uv - \int v \frac{du}{dx} dx$$

$$= \left[-xe^{-x}\right]_0^1 - \int_0^1 -e^{-x} dx$$

$$= \left[-1e^{-1}\right] - \left[-0e^0\right] + \int_0^1 e^{-x} dx$$

$$= -e^{-1} + \left[-e^{-x}\right]_0^1$$

$$= -e^{-1} + \left[-e^{-1}\right] - \left[-e^0\right]$$

$$= -e^{-1} - e^{-1} + 1$$

$$= 1 - 2e^{-1}$$

(b) $I_n = \int_0^1 x^n e^{-x} dx$

$$u = x^n \qquad \frac{dv}{dx} = e^{-x}$$

$$\Rightarrow \quad \frac{du}{dx} = nx^{n-1} \qquad v = -e^{-x}$$

$$I = uv - \int v \frac{du}{dx} dx$$

$$= \left[-x^n e^{-x}\right]_0^1 - \int_0^1 -nx^{n-1} e^{-x} dx$$

$$= \left[-1^n e^{-1}\right] - \left[-0^n e^0\right] + n\int_0^1 x^{n-1} e^{-x} dx$$

$$= -e^{-1} + nI_{n-1} \qquad [\text{since } 1^n = 1 \text{ and } 0^n = 0]$$

Hence $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.

(c) $I_1 = 1 - 2e^{-1}$ and $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$.

The recurrence relation $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$, along with the starting value $I_1 = 1 - 2e^{-1}$, can be used to generate the values of I_2, I_3, ...

Substituting $n = 2$ in the recurrence relation gives

$$\begin{aligned} I_2 &= 2I_1 - e^{-1} \\ &= 2(1 - 2e^{-1}) - e^{-1} \\ &= 2 - 4e^{-1} - e^{-1} \\ &= 2 - 5e^{-1} \end{aligned}$$

Substituting $n = 3$ in the recurrence relation gives

$$\begin{aligned} I_3 &= 3I_2 - e^{-1} \\ &= 3(2 - 5e^{-1}) - e^{-1} \\ &= 6 - 15e^{-1} - e^{-1} \\ &= 6 - 16e^{-1} \end{aligned}$$

[In the context of using a recurrence relation to generate values of definite integrals, the recurrence relation is often known as a **reduction formula.**]