ADVANCED HIGHER MATHEMATICS

DIFFERENTIAL EQUATIONS

A **differential equation** is an equation involving one or more derivatives.

Examples

(1) $(x^2-1)\frac{dy}{dx}+2xy=x$ is a **first-order differential equation**, as the equation contains a first derivative only.

(2) $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+4y=0$ is a **second-order differential equation**, as the equation contains a second derivative.

VERIFICATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

Given a function $y=f(x)$, you can verify that the function satisfies a particular differential equation.

Worked Example

Verify that the function $y=Ae^{5x}+Be^{-2x}$, where A and B are constants, satisfies the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}-10y=0.$$

Solution

$$y=Ae^{5x}+Be^{-2x} \quad\Rightarrow\quad \frac{dy}{dx}=5Ae^{5x}-2Be^{-2x}$$

$$\Rightarrow\quad \frac{d^2y}{dx^2}=25Ae^{5x}+4Be^{-2x}$$

$$\begin{aligned}&\frac{d^2y}{dx^2}-3\frac{dy}{dx}-10y\\&=25Ae^{5x}+4Be^{-2x}-3(5Ae^{5x}-2Be^{-2x})-10(Ae^{5x}+Be^{-2x})\\&=25Ae^{5x}+4Be^{-2x}-15Ae^{5x}+6Be^{-2x}-10Ae^{5x}-10Be^{-2x}\\&=0\end{aligned}$$

Hence $y=Ae^{5x}+Be^{-2x}$ satisfies the differential equation $\frac{d^2y}{dx^2}-3\frac{dy}{dx}-10y=0$.

YOU CAN NOW ATTEMPT THE WORKSHEET "VERIFICATION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS."

GENERAL SOLUTIONS OF DIFFERENTIAL EQUATIONS

Given a function $y = f(x)$, it is usually straightforward to verify that y satisfies a particular differential equation.

For example, we have already verified that the function $y = Ae^{5x} + Be^{-2x}$, where A and B are constants, satisfies the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = 0$.

Given a differential equation on its own, however, it is not so straightforward to find the general form of the function y which satisfies the differential equation. For example, given the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = 0$, how do we solve this differential equation to find that $y = Ae^{5x} + Be^{-2x}$ is the **general solution** of the differential equation.

FIRST-ORDER DIFFERENTIAL EQUATIONS WITH VARIABLES SEPARABLE

One approach which can sometimes be used to solve first-order differential equations is to **separate the variables** in the differential equation.

If a first-order differential equation can be written in the form

$$f(y)dy = g(x)dx,$$

where $f(y)$ is a function of y only and $g(x)$ is a function of x only, we say that we have separated the variables x and y.

The general solution of the differential equation is then found be integrating each sides with respect to the appropriate variable,

$$\text{i.e.} \quad \int f(y)dy = \int g(x)dx.$$

The following examples illustrate this.

Worked Example 1

Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y}.$$

Solution

$$\frac{dy}{dx} = \frac{x^2}{y} \qquad \Rightarrow \qquad y\,dy = x^2\,dx$$

The variables have been separated.

$$\int y\,dy = \int x^2\,dx$$

$$\Rightarrow \quad \frac{1}{2}y^2 = \frac{1}{3}x^3 + C \qquad [\times 6]$$

$$\Rightarrow \quad 3y^2 = 2x^3 + C$$

NOTES

(1) When solving differential equations, it is convenient for C to always denote the "total" constant so far in each line of working, although strictly speaking a different letter should be used for each new constant.

(2) It is not always necessary, or convenient, to express the general solution for y **explicitly** in terms of x; that is, in the form $y = f(x)$. Often it is more convenient, as in the above example, to express y **implicitly** in terms of x, for example in the form $y^2 = f(x)$ or $\sin y = f(x)$. Of course, the explicit solution must be given if requested.

Worked Example 2

Find the general solution of the differential equation

$$e^{4y}\frac{dy}{dx}-x=2.$$

Solution

$$e^{4y}\frac{dy}{dx}-x=2 \quad \Rightarrow \quad e^{4y}\frac{dy}{dx}=x+2$$

$$\Rightarrow \quad e^{4y}dy=(x+2)dx$$

The variables have been separated.

$$\int e^{4y}dy=\int(x+2)dx$$

$$\Rightarrow \quad \frac{1}{4}e^{4y}=\frac{1}{2}x^2+2x+C \qquad [\times 4]$$

$$\Rightarrow \quad e^{4y}=2x^2+8x+C$$

[If it is required to express y explicitly in terms of x, this can be done as follows:

$$e^{4y}=2x^2+8x+C \qquad [\ln]$$

$$\Rightarrow \quad 4y=\ln(2x^2+8x+C)$$

$$\Rightarrow \quad y=\frac{1}{4}\ln(2x^2+8x+C)\,]$$

Worked Example 3

Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{2}{\sin y}.$$

Solution

$$\frac{dy}{dx} = \frac{2}{\sin y} \qquad \Rightarrow \qquad \sin y dy = 2dx$$

The variables have been separated.

$$\int \sin y dy = \int 2dx$$

$\Rightarrow \quad -\cos y = 2x + C \qquad [\times(-1)]$

$\Rightarrow \quad \cos y = -2x + C$

$\Rightarrow \quad \cos y = C - 2x$

Worked Example 4

Find the general solution of the differential equation

$$\frac{dy}{dx} = 2x\sqrt{9 - y^2}\,.$$

Solution

$$\frac{dy}{dx} = 2x\sqrt{9 - y^2} \quad \Rightarrow \quad dy = 2x\sqrt{9 - y^2}\,dx$$

$$\Rightarrow \quad \frac{1}{\sqrt{9 - y^2}}\,dy = 2x\,dx$$

The variables have been separated.

$$\int \frac{1}{\sqrt{9 - y^2}}\,dy = \int 2x\,dx$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{y}{3}\right) = x^2 + C \qquad [\sin]$$

$$\Rightarrow \quad \frac{y}{3} = \sin(x^2 + C)$$

$$\Rightarrow \quad y = 3\sin(x^2 + C)$$

YOU CAN NOW ATTEMPT THE WORKSHEET "DIFFERENTIAL EQUATIONS 1."

In some cases the properties listed below of the logarithmic and exponential functions can be used to simplify general solutions of differential equations.

(1)	$e^{a+b} = e^a e^b$
(2)	$\ln a + \ln b = \ln(ab)$
(3)	$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$
(4)	$n \ln a = \ln(a^n)$

<u>Worked Example 5</u>

Find the general solution of the differential equation

$$\frac{dy}{dx} = 4y.$$

<u>Solution</u>

$$\frac{dy}{dx} = 4y \quad \Rightarrow \quad dy = 4y\,dx$$

$$\Rightarrow \quad \frac{1}{y}\,dy = 4\,dx$$

The variables have been separated.

$$\int \frac{1}{y}\,dy = \int 4\,dx$$

$\Rightarrow \quad \ln y = 4x + C \qquad$ [e^]

$\Rightarrow \quad y = e^{4x+C}$

$\Rightarrow \quad y = e^{4x} \cdot e^{C}$

$\Rightarrow \quad y = Ae^{4x} \qquad$ [where $A = e^C$]

Worked Example 6

Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y+2}{x+1}.$$

Solution

$$\frac{dy}{dx} = \frac{y+2}{x+1} \quad \Rightarrow \quad (x+1)dy = (y+2)dx$$

$$\Rightarrow \quad \frac{1}{y+2}dy = \frac{1}{x+1}dx$$

The variables have been separated.

$$\int \frac{1}{y+2}dy = \int \frac{1}{x+1}dx$$

$$\Rightarrow \quad \ln(y+2) = \ln(x+1) + C \quad [\text{e}^\wedge]$$

$$\Rightarrow \quad y+2 = (x+1)\cdot e^{C}$$

$$\Rightarrow \quad y+2 = A(x+1) \quad [\text{where } A = e^{C}]$$

$$\Rightarrow \quad y = A(x+1) - 2$$

Worked Example 7

Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{2x-1}.$$

Solution

$$\frac{dy}{dx} = \frac{y}{2x-1} \quad \Rightarrow \quad (2x-1)dy = ydx$$

$$\Rightarrow \quad \frac{1}{y}dy = \frac{1}{2x-1}dx$$

The variables have been separated.

$$\int \frac{1}{y}dy = \int \frac{1}{2x-1}dx$$

$$\Rightarrow \quad \ln y = \frac{1}{2}\ln(2x-1) + C$$

$$\Rightarrow \quad \ln y = \ln(2x-1)^{\frac{1}{2}} + C$$

$$\Rightarrow \quad \ln y = \ln\sqrt{2x-1} + C \qquad [e^\wedge]$$

$$\Rightarrow \quad y = \sqrt{2x-1}\cdot e^C$$

$$\Rightarrow \quad y = A\sqrt{2x-1} \qquad [\text{where } A = e^C]$$

Note that $\frac{1}{2}\ln(2x-1)$ should be written as $\ln\sqrt{2x-1}$ before taking the exponential of both sides of the equation.

YOU CAN NOW ATTEMPT QUESTION 1 OF THE WORKSHEET "DIFFERENTIAL EQUATIONS 2."

Worked Example 8

(a) Express $\dfrac{2x+3}{x(x+1)}$ in partial fractions.

(b) Hence find the general solution of the differential equation

$$x(x+1)\frac{dy}{dx} = y(2x+3),$$

expressing y explicitly in terms of x.

Solution

(a) It can be shown that $\dfrac{2x+3}{x(x+1)} = \dfrac{3}{x} - \dfrac{1}{x+1}$.

(b)
$$x(x+1)\frac{dy}{dx} = y(2x+3)$$
$$\Rightarrow \quad x(x+1)dy = y(2x+3)dx$$
$$\Rightarrow \quad \frac{1}{y}dy = \frac{2x+3}{x(x+1)}dx$$

The variables have been separated.

$$\int \frac{1}{y}dy = \int \frac{2x+3}{x(x+1)}dx$$
$$\Rightarrow \quad \int \frac{1}{y}dy = \int\left(\frac{3}{x} - \frac{1}{x+1}\right)dx$$
$$\Rightarrow \quad \ln y = 3\ln x - \ln(x+1) + C$$
$$\Rightarrow \quad \ln y = \ln(x^3) - \ln(x+1) + C$$
$$\Rightarrow \quad \ln y = \ln\left(\frac{x^3}{x+1}\right) + C \qquad [e^\wedge]$$
$$\Rightarrow \quad y = \left(\frac{x^3}{x+1}\right)\cdot e^C$$
$$\Rightarrow \quad y = \frac{Ax^3}{x+1} \qquad [\text{where } A = e^C]$$

Worked Example 9

(a) Express $\frac{x+1}{x(2x+1)}$ in partial fractions.

(b) Hence find the general solution of the differential equation

$$\frac{dy}{dx}=\frac{y(x+1)}{x(2x+1)},$$

expressing y explicitly in terms of x.

Solution

(a) It can be shown that $\frac{x+1}{x(2x+1)}=\frac{1}{x}-\frac{1}{2x+1}$.

(b)
$$\frac{dy}{dx}=\frac{y(x+1)}{x(2x+1)}$$
$$\Rightarrow \quad x(2x+1)dy=y(x+1)dx$$
$$\Rightarrow \quad \frac{1}{y}dy=\frac{x+1}{x(2x+1)}dx$$

The variables have been separated.

$$\int\frac{1}{y}dy=\int\frac{x+1}{x(2x+1)}dx$$
$$\Rightarrow \quad \int\frac{1}{y}dy=\int\left(\frac{1}{x}-\frac{1}{2x+1}\right)dx$$
$$\Rightarrow \quad \ln y=\ln x-\frac{1}{2}\ln(2x+1)+C$$
$$\Rightarrow \quad \ln y=\ln x-\ln(2x+1)^{\frac{1}{2}}+C$$
$$\Rightarrow \quad \ln y=\ln x-\ln\sqrt{2x+1}+C$$
$$\Rightarrow \quad \ln y=\ln\left(\frac{x}{\sqrt{2x+1}}\right)+C \quad [e^\wedge]$$
$$\Rightarrow \quad y=\left(\frac{x}{\sqrt{2x+1}}\right)\cdot e^C$$
$$\Rightarrow \quad y=\frac{Ax}{\sqrt{2x+1}} \quad [\text{where } A=e^C]$$

YOU CAN NOW ATTEMPT QUESTIONS 2 TO 18 OF THE WORKSHEET "DIFFERENTIAL EQUATION 2."

PARTICULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS

Worked Example 1

Find the particular solution of the differential equation

$$y^2 \frac{dy}{dx} = 4x^2 + 1,$$

given that $y = 2$ when $x = 1$, expressing y explicitly in terms of x.

Solution

$$y^2 \frac{dy}{dx} = 4x^2 + 1$$

$$\Rightarrow \quad y^2 dy = (4x^2 + 1)dx$$

The variables have been separated.

$$\int y^2 dy = \int (4x^2 + 1)dx$$

$$\Rightarrow \quad \frac{1}{3}y^3 = \frac{4}{3}x^3 + x + C \qquad [\times 3]$$

$$\Rightarrow \quad y^3 = 4x^3 + 3x + C$$

$y = 2$ when $x = 1$ $\Rightarrow$ $2^3 = 4 \times 1^3 + 3 \times 1 + C$

$\Rightarrow$ $7 + C = 8$

$\Rightarrow$ $C = 1$

Hence $y^3 = 4x^3 + 3x + 1$ $\Rightarrow$ $y = (4x^3 + 3x + 1)^{\frac{1}{3}}$

Worked Example 2

Find the particular solution of the differential equation

$$\frac{dy}{dx} = 2xy^2 ,$$

given that $y = -\frac{3}{10}$ when $x = -2$, expressing y explicitly in terms of x.

Solution

$$\frac{dy}{dx} = 2xy^2$$

$\Rightarrow \quad dy = 2xy^2 dx$

$\Rightarrow \quad \frac{1}{y^2} dy = 2x dx$

The variables have been separated.

$$\int y^{-2} dy = \int 2x dx$$

$\Rightarrow \quad -y^{-1} = x^2 + C$

$\Rightarrow \quad -\frac{1}{y} = x^2 + C \quad [\times(-1)]$

$\Rightarrow \quad \frac{1}{y} = -x^2 + C$

$\Rightarrow \quad y = \frac{1}{C - x^2}$

$y = -\frac{3}{10}$ when $x = -2$ $\quad \Rightarrow \quad -\frac{3}{10} = \frac{1}{C - (-2)^2}$

$\Rightarrow \quad -\frac{3}{10} = \frac{1}{C - 4}$

$\Rightarrow \quad -3(C - 4) = 10$

$\Rightarrow \quad -3C + 12 = 10$

$\Rightarrow \quad C = \frac{2}{3}$

Hence $y = \frac{1}{\frac{2}{3} - x^2} = \frac{3}{2 - 3x^2}$.

Worked Example 3

Find the particular solution of the differential equation

$$(x+4)\frac{dy}{dx}+1=y,$$

given that $y=7$ when $x=-1$, expressing y explicitly in terms of x.

Solution

$$(x+4)\frac{dy}{dx}+1=y$$

$$\Rightarrow \quad (x+4)\frac{dy}{dx}=y-1$$

$$\Rightarrow \quad (x+4)dy=(y-1)dx$$

$$\Rightarrow \quad \frac{1}{y-1}dy=\frac{1}{x+4}dx$$

The variables have been separated.

$$\int\frac{1}{y-1}dy=\int\frac{1}{x+4}dx$$

$$\Rightarrow \quad \ln(y-1)=\ln(x+4)+C \qquad [e^\wedge]$$

$$\Rightarrow \quad y-1=(x+4)\cdot e^C$$

$$\Rightarrow \quad y-1=A(x+4) \qquad [\text{where } A=e^C]$$

$$\Rightarrow \quad y=A(x+4)+1$$

$y=7$ when $x=-1$ $\Rightarrow$ $7=A(3)+1$

$\Rightarrow$ $3A+1=7$

$\Rightarrow$ $A=2$

Hence $y=2(x+4)+1$ $\Rightarrow$ $y=2x+9$

Worked Example 4

(a) Express $\dfrac{2}{(y+1)(y+3)}$ in partial fractions.

(b) Hence find the particular solution of the differential equation

$$2x\frac{dy}{dx} = (y+1)(y+3),$$

given that $y = -\dfrac{5}{3}$ when $x = -1$, expressing y explicitly in terms of x.

Solution

(a) It can be shown that $\dfrac{2}{(y+1)(y+3)} = \dfrac{1}{y+1} - \dfrac{1}{y+3}$.

(b)

$$2x\frac{dy}{dx} = (y+1)(y+3)$$

$$\Rightarrow \quad 2xdy = (y+1)(y+3)dy$$

$$\Rightarrow \quad \frac{2}{(y+1)(y+3)}dy = \frac{1}{x}dx$$

The variables have been separated.

$$\int \frac{2}{(y+1)(y+3)}dy = \int \frac{1}{x}dx$$

$$\Rightarrow \quad \int \left(\frac{1}{y+1} - \frac{1}{y+3}\right)dy = \int \frac{1}{x}dx$$

$$\Rightarrow \quad \ln(y+1) - \ln(y+3) = \ln x + C$$

$$\Rightarrow \quad \ln\left(\frac{y+1}{y+3}\right) = \ln x + C \qquad [e^\wedge]$$

$$\Rightarrow \quad \frac{y+1}{y+3} = x \cdot e^C$$

$$\Rightarrow \quad \frac{y+1}{y+3} = Ax \qquad [A = e^C]$$

$$\Rightarrow \quad y+1 = Ax(y+3)$$

$$\Rightarrow \quad y+1 = Axy + 3Ax$$

$$\Rightarrow \quad y - Axy = 3Ax - 1$$

$$\Rightarrow \quad y(1-Ax) = 3Ax - 1$$

$$\Rightarrow \quad y = \frac{3Ax-1}{1-Ax}$$

$$y = -\frac{5}{3} \text{ when } x = -1 \Rightarrow \quad -\frac{5}{3} = \frac{3A(-1)-1}{1-A(-1)}$$

$$\Rightarrow \quad -\frac{5}{3} = \frac{-3A-1}{1+A}$$

$$\Rightarrow \quad -5(1+A) = 3(-3A-1)$$

$$\Rightarrow \quad -5-5A = -9A-3$$

$$\Rightarrow \quad 4A = 2$$

$$\Rightarrow \quad A = \frac{1}{2}$$

Hence $y = \dfrac{\frac{3}{2}x - 1}{1 - \frac{1}{2}x} = \dfrac{3x-2}{2-x}$.

[Alternatively the value of A can be found by substituting $y = -\frac{5}{3}$ and $x = -1$ into the equation $\dfrac{y+1}{y+3} = Ax$.

Then $\dfrac{-\frac{5}{3}+1}{-\frac{5}{3}+3} = A(-1) \quad \Rightarrow \quad \dfrac{-5+3}{-5+9} = -A$

$$\Rightarrow \quad \frac{-2}{4} = -A$$

$$\Rightarrow \quad A = \frac{1}{2}$$

$$\frac{y+1}{y+3} = Ax \quad \Rightarrow \quad \frac{y+1}{y+3} = \frac{1}{2}x$$

$$\Rightarrow \quad \frac{y+1}{y+3} = \frac{x}{2}$$

$$\Rightarrow \quad 2(y+1) = x(y+3)$$

$$\Rightarrow \quad 2y+2 = xy+3x$$

$$\Rightarrow \quad 2y - xy = 3x-2$$

$$\Rightarrow \quad y(2-x) = 3x-2$$

$$\Rightarrow \quad y = \frac{3x-2}{2-x}$$]

YOU CAN NOW ATTEMPT THE WORKSHEET
"PARTICULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS."

DIFFERENTIAL EQUATIONS AS MATHEMATICAL MODELS

Worked Example 1

The number of strands of bacteria, x, present in a culture after t days of growth is assumed to be increasing at a rate proportional to the number of strands present.

(a) Write down a differential equation which represents this and find the general solution for x in terms of t.

(b) Given that there are 326 strands initially present and that the number of strands observed after 4 days is 1833, estimate the number of strands likely to be present after one week.

Solution

(a) The rate of growth of the bacteria is $\frac{dx}{dt}$ and the number of strands present is x.

Hence $\frac{dx}{dt}$ is proportional to x $\Rightarrow$ $\frac{dx}{dt} = kx$, where k is a constant.

$$\frac{dx}{dt} = kx \quad \Rightarrow \quad dx = kxdt$$

$$\Rightarrow \quad \frac{1}{x}dx = kdt$$

The variables have been separated.

$$\int \frac{1}{x}dx = \int kdt \quad \Rightarrow \quad \ln x = kt + C \quad [e\text{^}]$$

$$\Rightarrow \quad x = e^{kt+C}$$

$$\Rightarrow \quad x = e^{kt} \cdot e^{C}$$

$$\Rightarrow \quad x = Ae^{kt}$$

(b) When $t = 0$, $x = 326$ $\Rightarrow$ $326 = Ae^{k\times 0}$

$$\Rightarrow \quad Ae^{0} = 326$$

$$\Rightarrow \quad A = 326$$

When $t = 4$, $x = 1833$ $\Rightarrow$ $1833 = Ae^{k\times 4}$

$$\Rightarrow \quad 326e^{4k} = 1833$$

$$\Rightarrow \quad e^{4k} = \frac{1833}{326} \quad [\ln]$$

$$\Rightarrow \quad 4k = \ln\left(\frac{1833}{326}\right)$$

$$\Rightarrow \qquad k = \frac{1}{4}\ln\left(\frac{1833}{326}\right) = 0\cdot4317$$

Hence $x = 326e^{0\cdot4317t}$.

When $t = 7$: $\quad x = 326e^{0\cdot4317\times7} = 326e^{3\cdot0219} = 6693$

About 6693 strands are likely to be present after one week.

Worked Example 2

According to Newton's law of cooling, the rate at which an object cools is proportional to the difference in temperature between the object and the surrounding medium. When an object cools surrounded by air at a temperature of 75 °C, the cooling of the object is governed by a differential equation of the form

$$\frac{dT}{dt} = -k(T - 75),$$

where T °C is the temperature of the object after t hours of cooling and k is a constant.

(a) Find the general solution of this differential equation, expressing T as a function of t.

(b) Given that a certain object cools from 125 °C to 100 °C in half an hour when surrounded by air at a temperature of 75 °C, find:

(i) the temperature of this object at the end of another half-hour
(ii) the time taken for the temperature of this object to fall to 80 °C.

Solution

(a) $\frac{dT}{dt} = -k(T - 75) \quad \Rightarrow \quad dT = -k(T - 75)dt$

$\Rightarrow \quad \frac{1}{T - 75}dT = -kdt$

The variables have been separated.

$\int \frac{1}{T-75}dT = \int -kdt \Rightarrow \quad \ln(T - 75) = -kt + C \quad$ [e^]

$\Rightarrow \quad T - 75 = e^{-kt+C}$

$\Rightarrow \quad T - 75 = e^{-kt} \cdot e^{C}$

$\Rightarrow \quad T - 75 = Ae^{-kt} \quad$ [where $A = e^{C}$]

$\Rightarrow \quad T = 75 + Ae^{-kt}$

(b) When $t = 0$, $T = 125 \Rightarrow \quad 125 = 75 + Ae^{-k \times 0}$

$\Rightarrow \quad 75 + Ae^{0} = 125$

$\Rightarrow \quad 75 + A = 125$

$\Rightarrow \quad A = 50$

When $t = \frac{1}{2}$, $T = 100 \Rightarrow \quad 100 = 75 + Ae^{-k \times \frac{1}{2}}$

$\Rightarrow \quad Ae^{-\frac{1}{2}k} = 25$

$\Rightarrow \quad 50e^{-\frac{1}{2}k} = 25$

$$\Rightarrow \quad e^{-\frac{1}{2}k} = \frac{1}{2} \quad [\ln]$$

$$\Rightarrow \quad -\frac{1}{2}k = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad k = -2\ln\left(\frac{1}{2}\right) = 1 \cdot 3863$$

Hence $T = 75 + 50e^{-1\cdot 3863t}$.

(i) When $t = 1$: $T = 75 + 50e^{-1\cdot 3863\times 1}$
$= 75 + 50e^{-1\cdot 3863}$
$= 87 \cdot 5$

The temperature of the object if 87·5 °C at the end of another half hour.

(ii) When $T = 80$: $80 = 75 + 50e^{-1\cdot 3863t}$

$$\Rightarrow \quad 50e^{-1\cdot 3863t} = 5$$

$$\Rightarrow \quad e^{-1\cdot 3863t} = \frac{1}{10} \quad [\ln]$$

$$\Rightarrow \quad -1 \cdot 3863t = \ln\left(\frac{1}{10}\right)$$

$$\Rightarrow \quad t = \frac{\ln\left(\frac{1}{10}\right)}{-1 \cdot 3863} = 1 \cdot 66$$

It will take 1·66 hours for the temperature of the object to fall to 80 °C.

Worked Example 3

When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water. This can be represented by the differential equation

$$\frac{dh}{dt} = -k\sqrt{h},$$

where h metres is the depth of the water t minutes after the valve was opened and k is a positive constant.

(a) Find the general solution of the differential equation. (You need **not** express h explicitly as a function of t.)

(b) Given that the pool was initially 9 metres deep and that the depth of water was 4 metres after 20 minutes of draining, how long did it take to drain the pool completely?

Solution

(a) $\frac{dh}{dt} = -k\sqrt{h} \quad \Rightarrow \quad dh = -k\sqrt{h}dt$

$\Rightarrow \quad \frac{1}{\sqrt{h}}dh = -kdt$

The variables have been separated.

$\int h^{-\frac{1}{2}}dh = \int -kdt \quad \Rightarrow \quad 2h^{\frac{1}{2}} = -kt + C$

$\Rightarrow \quad 2\sqrt{h} = C - kt$

(b) When $t = 0$, $h = 9 \quad \Rightarrow \quad 2\sqrt{9} = C - k \times 0$

$\Rightarrow \quad C = 6$

When $t = 20$, $h = 4 \quad \Rightarrow \quad 2\sqrt{4} = C - k \times 20$

$\Rightarrow \quad 6 - 20k = 4$

$\Rightarrow \quad 20k = 2$

$\Rightarrow \quad k = \frac{1}{10}$

Hence $2\sqrt{h} = 6 - \frac{1}{10}t$, i.e. $2\sqrt{h} = 6 - \frac{t}{10}$.

When $h = 0$: $$2\sqrt{0} = 6 - \frac{t}{10}$$

$$\Rightarrow \quad 0 = 6 - \frac{t}{10}$$

$$\Rightarrow \quad \frac{t}{10} = 6$$

$$\Rightarrow \quad t = 60$$

It takes 60 minutes (i.e. 1 hour) to drain the pool completely.

Worked Example 4

In a chemical reaction, two substances X and Y combine to form a third substance Z.

Let Q denote the number of grams of Z formed t minutes after the reaction begins. The rate at which Q varies is represented by the differential equation

$$\frac{dQ}{dt} = \frac{(20-Q)(10-Q)}{400}.$$

(a) Express $\dfrac{400}{(20-Q)(10-Q)}$ in partial fractions.

(b) Given that $Q=0$ when $t=0$, find the general solution of the differential equation, expressing Q explicitly as a function of t.

(c) Find:

(i) the number of grams of Z formed one hour after the reaction begins
(ii) the time taken to form 5 grams of Z.

(d) Find the limiting value of Q as $t \to \infty$.

Solution

(a) It can be shown that $\dfrac{400}{(20-Q)(10-Q)} = -\dfrac{40}{20-Q} + \dfrac{40}{10-Q}$,

(b) $\dfrac{dQ}{dt} = \dfrac{(20-Q)(10-Q)}{400} \quad \Rightarrow \quad 400dQ = (20-Q)(10-Q)dt$

$$\Rightarrow \quad \frac{400}{(20-Q)(10-Q)}dQ = dt$$

The variables have been separated.

$$\int \frac{400}{(20-Q)(10-Q)}dQ = \int dt$$

$$\Rightarrow \quad \int \left(-\frac{40}{20-Q} + \frac{40}{10-Q} \right) dQ = \int dt$$

$$\Rightarrow \quad -40\{-\ln(20-Q)\} + 40\{-\ln(10-Q)\} = t + C$$

$$\Rightarrow \quad 40\ln(20-Q) - 40\ln(10-Q) = t + C$$

$$\Rightarrow \quad 40\{\ln(20-Q) - \ln(10-Q)\} = t + C$$

$$\Rightarrow \quad 40\ln\left(\frac{20-Q}{10-Q} \right) = t + C$$

$$\Rightarrow \quad \ln\left(\frac{20-Q}{10-Q}\right) = \frac{t}{40} + C \quad [e^\wedge]$$

$$\Rightarrow \quad \frac{20-Q}{10-Q} = e^{\frac{t}{40}+C}$$

$$\Rightarrow \quad \frac{20-Q}{10-Q} = e^{\frac{t}{40}} \cdot e^{C}$$

$$\Rightarrow \quad \frac{20-Q}{10-Q} = Ae^{\frac{t}{40}} \quad [\text{where } A = e^{C}]$$

$Q = 0$ when $t = 0$ $\Rightarrow$ $\frac{20-0}{10-0} = Ae^{0}$

$\Rightarrow$ $A = 2$

Hence $\frac{20-Q}{10-Q} = 2e^{\frac{t}{40}}$

$$\Rightarrow \quad 20 - Q = 2e^{\frac{t}{40}}(10-Q)$$

$$\Rightarrow \quad 20 - Q = 20e^{\frac{t}{40}} - 2Qe^{\frac{t}{40}}$$

$$\Rightarrow \quad 2Qe^{\frac{t}{40}} - Q = 20e^{\frac{t}{40}} - 20$$

$$\Rightarrow \quad Q\left(2e^{\frac{t}{40}} - 1\right) = 20\left(e^{\frac{t}{40}} - 1\right)$$

$$\Rightarrow \quad Q = \frac{20\left(e^{\frac{t}{40}} - 1\right)}{2e^{\frac{t}{40}} - 1}$$

(c) (i) When $t = 60$: $Q = \frac{20\left(e^{\frac{60}{40}} - 1\right)}{2e^{\frac{60}{40}} - 1} = \frac{20(e^{1\cdot 5} - 1)}{2e^{1\cdot 5} - 1} = 8\cdot 74$

8·74 grams of Z are formed on hour after the reaction begins.

(ii) When $Q = 5$: $5 = \frac{20\left(e^{\frac{t}{40}} - 1\right)}{2e^{\frac{t}{40}} - 1}$

$$\Rightarrow \quad 5\left(2e^{\frac{t}{40}} - 1\right) = 20\left(e^{\frac{t}{40}} - 1\right)$$

$$\Rightarrow \quad 10e^{\frac{t}{40}} - 5 = 20e^{\frac{t}{40}} - 20$$

$$\Rightarrow \quad 10e^{\frac{t}{40}} = 15$$

$$\Rightarrow \quad e^{\frac{t}{40}} = 1\cdot 5 \quad [\ln]$$

$$\Rightarrow \quad \frac{t}{40} = \ln 1\cdot 5$$

$$\Rightarrow \quad t = 40\ln 1\cdot 5 = 16\cdot 2$$

It takes 16·2 minutes to form 5 grams of Z.

(d) $$Q = \frac{20\left(e^{\frac{t}{40}} - 1\right)}{2e^{\frac{t}{40}} - 1}$$

As $t \to \infty$, both the numerator and denominator of the above expression will tend to infinity (since $e^{\frac{t}{40}} \to \infty$) and the limiting value of Q cannot be found using this expression. An alternative expression for Q must therefore be found before investigating the limiting value.

Dividing all terms in the above expression by $e^{\frac{t}{40}}$:

$$Q = \frac{20\left(1 - \frac{1}{e^{\frac{t}{40}}}\right)}{2 - \frac{1}{e^{\frac{t}{40}}}} = \frac{20\left(1 - e^{-\frac{t}{40}}\right)}{2 - e^{-\frac{t}{40}}}$$

Now as $t \to \infty$, $e^{-\frac{t}{40}} \to 0$ and $Q \to \dfrac{20(1-0)}{2-0} = 10$.

Hence the limiting value of Q as $t \to \infty$ is 10.

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