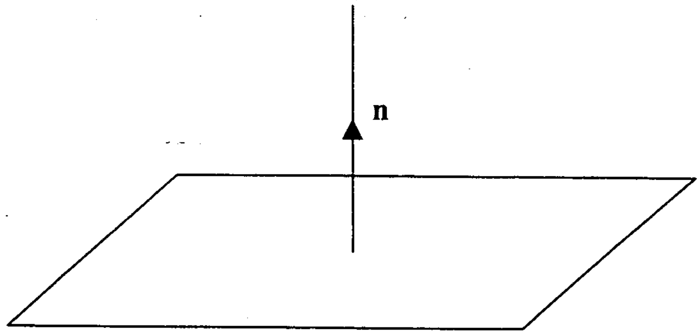
ADVANCED HIGHER MATHEMATICS

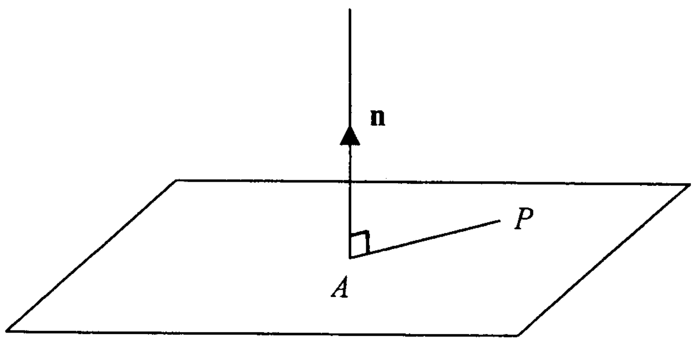
THE EQUATION OF A PLANE

A **plane** is simply a flat two dimensional surface.



A vector which is **perpendicular** to a plane is known as a **normal vector** and is denoted by **n**. [A normal vector is in fact perpendicular to all vectors in the plane.]

Consider the plane π in three dimensional space relative to a set of coordinate axes. Let A be a fixed point in this plane and let $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a vector normal to the plane passing through A. Let $P(x, y, z)$ be a typical point in the plane, as shown in the diagram below.



The vector $\overrightarrow{AP}$ is perpendicular to the normal vector **n**, hence $\mathbf{n} \bullet \overrightarrow{\mathbf{AP}} = 0$.

Now $\mathbf{n} \bullet \overrightarrow{\mathbf{AP}} = 0 \quad \Rightarrow \quad \mathbf{n} \bullet (\mathbf{p} - \mathbf{a}) = 0$

$\Rightarrow \quad \mathbf{n} \bullet \mathbf{p} - \mathbf{n} \bullet \mathbf{a} = 0$

$\Rightarrow \quad \mathbf{n} \bullet \mathbf{p} = \mathbf{n} \bullet \mathbf{a}$

Now $\mathbf{n} \bullet \mathbf{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$ and $\mathbf{n} \bullet \mathbf{a}$ is a constant since both the vector **n** and the point A are fixed.

If we let $\mathbf{n} \bullet \mathbf{a} = k$ (a constant), then the equation of the plane can be expressed as $ax + by + cz = k$.

SUMMARY

The equation of the plane with normal vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is of the form

$$ax + by + cz = k$$

where k is a constant.

The coordinates of any point $P(x, y, z)$ in the plane will satisfy this equation. The value of the constant k can be determined by substituting the coordinates of a point lying in the plane.

Worked Example 1

Find the equation of the plane perpendicular to the vector $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and containing the point $P(-1, 2, 1)$.

Solution

The vector $\mathbf{n} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ is a vector normal to the plane.

The equation of the plane is therefore of the form $x - 3y + 2z = k$, where k is a constant.

The point $P(-1, 2, 1)$ lies in the plane.

Substitute $x = -1$, $y = 2$ and $z = 1$:

$$x - 3y + 2z = k$$
$$\Rightarrow \quad -1 - 3 \times 2 + 2 \times 1 = k$$
$$\Rightarrow \quad k = -5$$

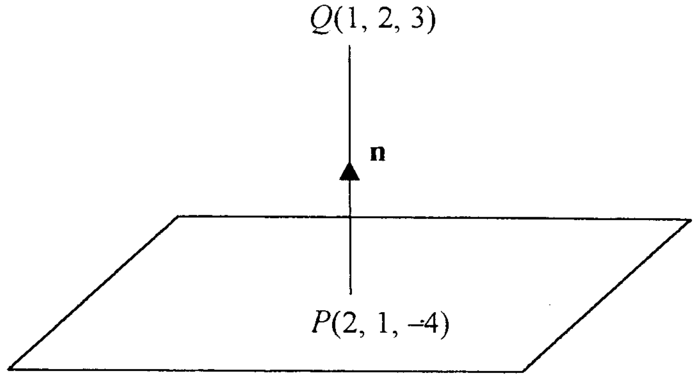
The equation of the plane is $x - 3y + 2z = -5$.

Worked Example 2

P is the point $(2,\ 1,\ -4)$ and Q is $(1,\ 2,\ 3)$.

Find the equation of the plane perpendicular to PQ which contains the point P.

Solution



A vector normal to the plane is

$$\mathbf{n} = \overrightarrow{PQ} = \begin{pmatrix} 1-2 \\ 2-1 \\ 3-(-4) \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 7 \end{pmatrix}$$

The equation of the plane is therefore of the form $-x + y + 7z = k$, where k is a constant.

The point $P(2,\ 1,\ -4)$ lies in the plane.

Substitute $x = 2$, $y = 1$ and $z = -4$:

$$-x + y + 7z = k$$
$$\Rightarrow \quad -2 + 1 + 7 \times (-4) = k$$
$$\Rightarrow \quad k = -29$$

The equation of the plane is $-x + y + 7z = -29$.

[*Note that the equation of the plane can also be expressed as* $x - y - 7z = 29$.
Note also that the coordinates of Q cannot be substituted into the equation of the plane since the point Q does not lie in the plane.]

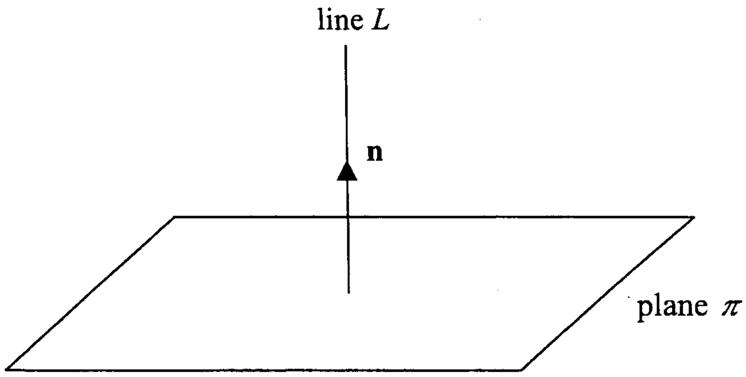
Worked Example 3

The equation of a line L is given by

$$\frac{x}{-2}=\frac{y+2}{-1}=\frac{z-9}{2}.$$

The plane π is perpendicular to the line L and passes through the point $(1, -4, 2)$. Find the equation of the plane π.

Solution



The vector $\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ is parallel to the line L.

Hence $\mathbf{n}=\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ is a vector normal to plane π.

The equation of plane π is therefore of the form $-2x-y+2z=k$, where k is a constant.

The point $(1, -4, 2)$ lies in the plane π.

Substitute $x=1$, $y=-4$ and $z=2$:

$$\begin{aligned} & -2x-y+2z=k \\ \Rightarrow\quad & -2\times1-(-4)+2\times2=k \\ \Rightarrow\quad & k=6 \end{aligned}$$

The equation of plane π is $-2x-y+2z=6$ (or $2x+2y-2z=-6$).

YOU CAN NOW ATTEMPT QUESTIONS 1 TO 3 OF THE WORKSHEET "THE EQUATION OF A PLANE".

THE VECTOR PRODUCT

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

Recall that the **scalar product** $\mathbf{a} \bullet \mathbf{b}$ is given by

$$\mathbf{a} \bullet \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

Note that $\mathbf{a} \bullet \mathbf{b}$ is a scalar (number), not a vector.

The vectors **a** and **b** can also be multiplied to give a vector as an answer. This is known as the **vector product** $\mathbf{a} \times \mathbf{b}$.

The vector product $\mathbf{a} \times \mathbf{b}$ is found as follows:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= (a_2 b_3 - a_3 b_2)\,\mathbf{i} - (a_1 b_3 - a_3 b_1)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k} \end{aligned}$$

There is no need to attempt to memorise this formula.

The formation of the vector product can be broken down into the following simple steps.

STEP 1

Write the vector product in the component form $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

STEP 2

Cover up the top row and calculate the determinant of the 2×2 "matrix" visible. This gives the coefficient of **i** in the vector product.

STEP 3

Cover up the middle row and calculate the determinant of the 2×2 "matrix" visible. This gives the coefficient of $-\mathbf{j}$ in the vector product.

STEP 4

Cover up the bottom row and calculate the determinant of the 2×2 "matrix" visible. This gives the coefficient of **k** in the vector product.

Example 1

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\1\\4\end{pmatrix}$$
$$=\mathbf{i}(2\times 4-1\times 3)-\mathbf{j}(1\times 4-2\times 3)+\mathbf{k}(1\times 1-2\times 2)$$
$$=\mathbf{i}(5)-\mathbf{j}(-2)+\mathbf{k}(-3)$$
$$=5\mathbf{i}+2\mathbf{j}-3\mathbf{k}$$

Worked Example 2

If $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + \mathbf{k}$, find the vector product $\mathbf{a} \times \mathbf{b}$.

Solution

$$\mathbf{a}\times\mathbf{b}=\begin{pmatrix}3\\-1\\2\end{pmatrix}\times\begin{pmatrix}0\\2\\1\end{pmatrix}$$
$$=\mathbf{i}((-1)\times 1-2\times 2)-\mathbf{j}(3\times 1-0\times 2)+\mathbf{k}(3\times 2-0\times(-1))$$
$$=\mathbf{i}(-5)\ -\mathbf{j}(3)\ +\mathbf{k}(6)$$
$$=-5\mathbf{i}-3\mathbf{j}+6\mathbf{k}$$

Worked Example 3

$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = 4\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

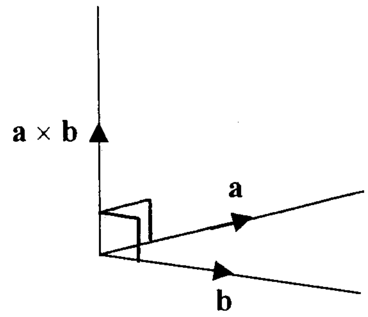
Find $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Solution

$$\mathbf{b}\times\mathbf{c}=\begin{pmatrix}1\\-1\\-2\end{pmatrix}\times\begin{pmatrix}4\\-3\\4\end{pmatrix}$$
$$=\mathbf{i}(-1\times 4-(-3)\times(-2))-\mathbf{j}(1\times 4-4\times(-2))+\mathbf{k}(1\times(-3)-4\times(-1))$$
$$=\mathbf{i}(-10)\ -\mathbf{j}(12)\ +\mathbf{k}(1)$$
$$=-10\mathbf{i}-12\mathbf{j}+\mathbf{k}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{pmatrix}3\\2\\-1\end{pmatrix}\times\begin{pmatrix}-10\\-12\\1\end{pmatrix}$$
$$=\mathbf{i}(2\times 1-(-12)\times(-1))-\mathbf{j}(3\times 1-(-10)\times(-1))+\mathbf{k}(3\times(-12)-(-10)\times 2)$$
$$=\mathbf{i}(-10)\ -\mathbf{j}(-7)\ +\mathbf{k}(-16)$$
$$=-10\mathbf{i}+7\mathbf{j}-16\mathbf{k}$$

An important property of the vector product is that the vector $\mathbf{a} \times \mathbf{b}$ is always perpendicular to each of the vectors **a** and **b**. We will make use of this fact shortly to find the equation of a plane.



[Note that the vector product $\mathbf{a} \times \mathbf{b}$ is **not equal** to the vector product $\mathbf{b} \times \mathbf{a}$.
In fact, $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.]

Worked Example 1

A plane is parallel to each of the vectors $\mathbf{a} = 4\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and contains the point (3, 4, –7).

Find the equation of the plane.

Solution

The vector $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ will be a vector normal to the plane.

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$$

$$= \mathbf{i}(0 \times 3 - (-2) \times (-1)) - \mathbf{j}(4 \times 3 - 6 \times (-1)) + \mathbf{k}(4 \times (-2) - 6 \times 0)$$

$$= \mathbf{i}(-2) - \mathbf{j}(18) + \mathbf{k}(-8)$$

$$= -2\mathbf{i} - 18\mathbf{j} - 8\mathbf{k}$$

The equation of the plane will therefore be of the form $-2x - 18y - 8z = k$, where k is a constant.

The point (3, 4, –7) lies in the plane.

Substitute $x = 3$, $y = 4$ and $z = -7$:

$$-2x - 18y - 8z = k$$

$$\Rightarrow \quad -2 \times 3 - 18 \times 4 - 8 \times (-7) = k$$

$$\Rightarrow \quad k = -22$$

The equation of the plane is $-2x - 18y - 8z = -22$ (or $2x + 18y + 8z = 22$ or $x + 9y + 4z = 11$).

Worked Example 2

Find the equation of the plane which contains the points $A(-2, 1, 2)$, $B(0, 2, 5)$ and $C(2, -1, 3)$.

Solution

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ both lie in the plane.

$$\overrightarrow{AB} = \begin{pmatrix} 0-(-2) \\ 2-1 \\ 5-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 2-(-2) \\ -1-1 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

The vector $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ will be a vector normal to the plane.

$$\begin{aligned} \mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \\ &= \mathbf{i}(1\times1-(-2)\times3)-\mathbf{j}(2\times1-4\times3)+\mathbf{k}(2\times(-2)-4\times1) \\ &= \mathbf{i}(7) - \mathbf{j}(-10) + \mathbf{k}(-8) \\ &= 7\mathbf{i} + 10\mathbf{j} - 8\mathbf{k} \end{aligned}$$

The equation of the plane will therefore be of the form $7x + 10y - 8z = k$, where k is a constant.

The point $A(-2, 1, 2)$ lies in the plane.

Substitute $x = -2$, $y = 1$ and $z = 2$:

$$\begin{aligned} & 7x + 10y - 8z = k \\ \Rightarrow \quad & 7\times(-2) + 10\times1 - 8\times2 = k \\ \Rightarrow \quad & k = -20 \end{aligned}$$

The equation of the plane is $7x + 10y - 8z = -20$.

[*Note that a different pair of vectors could have been used to find a vector normal to the plane, e.g.* $\overrightarrow{AB}$ *and* $\overrightarrow{BC}$, *or* $\overrightarrow{AC}$ *and* $\overrightarrow{BC}$.
Also, the coordinates of the points B or C could have been used to find the value of k.]

YOU CAN NOW ATTEMPT QUESTIONS 4 TO 15 OF THE WORKSHEET "THE EQUATION OF A PLANE".

THE ANGLE BETWEEN TWO PLANES

Let $\mathbf{n}_1$ be a vector normal to the plane π_1 and let $\mathbf{n}_2$ be a vector normal to the plane π_2.

Then it can be shown that the angle, θ, between the planes π_1 and π_2 is the angle between the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$:

$$\cos\theta = \frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Worked Example

The plane π_1 has equation $2x+3y+z=5$ and the plane π_2 has equation $x+y-z=0$.

Calculate the size of the acute angle between the planes π_1 and π_2.

Solution

π_1: $2x+3y+z=5$

The vector $\mathbf{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ is a vector normal to the plane π_1.

π_2: $x+y-z=0$

The vector $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is a vector normal to the plane π_2.

The angle, θ, between the planes π_1 and π_2 is the angle between the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$.

$$\begin{aligned}\cos\theta &= \frac{\mathbf{n}_1 \bullet \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \\ &= \frac{2\times 1+3\times 1+1\times(-1)}{\sqrt{2^2+3^2+1^2}\sqrt{1^2+1^2+(-1)^2}} \\ &= \frac{4}{\sqrt{14}\sqrt{3}} \\ &= 0\cdot 617...\end{aligned}$$

$\Rightarrow \quad \theta = \cos^{-1} 0\cdot 617... = 51\cdot 9°$

The acute angle between planes π_1 and π_2 is $51\cdot 9°$.

YOU CAN NOW ATTEMPT QUESTIONS 16 TO 20 OF THE WORKSHEET "THE EQUATION OF A PLANE".

THE POINT OF INTERSECTION OF A LINE AND A PLANE

The point of intersection of a line and a plane can be found when the equation of the line is expressed in parametric form.

Worked Example

Find the point of intersection of the line

$$\frac{x-3}{4}=\frac{y-2}{-1}=\frac{z+1}{2}$$

and the plane with equation $2x+y-z=4$.

Solution

The equation of the line must firstly be expressed in parametric form.

Line: $\frac{x-3}{4}=\frac{y-2}{-1}=\frac{z+1}{2}=t$

The equation of the line in parametric form is

$$x=4t+3, \quad y=-t+2, \quad z=2t-1$$

The line intersects the plane when

$$\begin{aligned}
& 2x+y-z=4 \\
\Rightarrow\quad & 2(4t+3)+(-t+2)-(2t-1)=4 \\
\Rightarrow\quad & 8t+6-t+2-2t+1=4 \\
\Rightarrow\quad & 5t+9=4 \\
\Rightarrow\quad & t=-1
\end{aligned}$$

The coordinates of the point of intersection can now be found by substitution:

$$\begin{aligned}
x &= 4t+3=4\times(-1)+3=-1 \\
y &= -t+2=-(-1)+2=3 \\
z &= 2t-1=2\times(-1)-1=-3
\end{aligned}$$

The point of intersection is (–1, 3, –3).

THE ANGLE BETWEEN A LINE AND A PLANE

Suppose that the line L intersects the plane π.

Let **a** be a vector in the direction of the line L and let **n** be a vector normal to the plane π.

Then it can be shown that the angle, θ, between the line and the plane is given by

$$\sin\theta = \frac{\mathbf{a} \bullet \mathbf{n}}{|\mathbf{a}||\mathbf{n}|}$$

Note that this is an equation for $\sin\theta$.

Worked Example

Calculate the size of the acute angle between the line

$$\frac{x-3}{4}=\frac{y-2}{-1}=\frac{z+1}{2}$$

and the plane with equation $2x+y-z=4$.

Solution

The vector $\mathbf{a}=\begin{pmatrix}4\\-1\\2\end{pmatrix}$ is a vector in the direction of the line.

The vector $\mathbf{n}=\begin{pmatrix}2\\1\\-1\end{pmatrix}$ is a vector normal to the plane.

The angle, θ, between the line and the plane is given by

$$\begin{aligned}\sin\theta &= \frac{\mathbf{a}\bullet\mathbf{n}}{|\mathbf{a}||\mathbf{n}|}\\ &= \frac{4\times 2+(-1)\times 1+2\times(-1)}{\sqrt{4^2+(-1)^2+2^2}\sqrt{2^2+1^2+(-1)^2}}\\ &= \frac{5}{\sqrt{21}\sqrt{6}}\\ &= 0\cdot 445...\end{aligned}$$

$\Rightarrow \quad \theta=\sin^{-1}0\cdot 445...=26\cdot 5°$

The acute angle between the line and the plane is $26\cdot 5°$.

THE LINE OF INTERSECTION OF TWO PLANES

Given that two planes intersect, the planes will intersect in a line. The equation of the line of intersection of two planes can be found in parametric form using the method illustrated in the following example.

Worked Example

The equations of two planes are $x-4y+2z=1$ and $x-y-z=-5$. By putting $z=t$, obtain parametric equations for the line of intersection of the planes.

Solution

Plane π_1: $\quad x-4y+2z=1$

Put $z=t$ $\quad \Rightarrow \quad x-4y+2t=1$

$\Rightarrow \quad x-4y=1-2t \quad \ldots(1)$

Plane π_2: $\quad x-y-z=-5$

Put $z=t$ $\quad \Rightarrow \quad x-y-t=-5$

$\Rightarrow \quad x-y=-5+t \quad \ldots(2)$

Equations (1) and (2) can now be solved simultaneously to express *x* and *y* in terms of the parameter *t*. Note that *x* can be eliminated by subtracting the equations.

$$\begin{aligned}
(2)-(1) \quad &\Rightarrow \quad (x-y)-(x-4y)=(-5+t)-(1-2t) \\
&\Rightarrow \quad x-y-x+4y=-5+t-1+2t \\
&\Rightarrow \quad 3y=-6+3t \\
&\Rightarrow \quad y=-2+t
\end{aligned}$$

Sub. in (2): $\quad x-y=-5+t \Rightarrow \quad x=-5+t+y$

$\Rightarrow \quad x=-5+t-2+t$

$\Rightarrow \quad x=-7+2t$

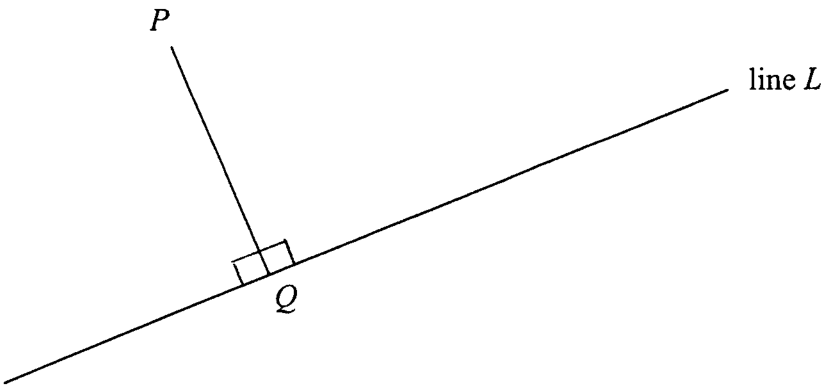
The parametric equations of the line of intersection of the plane are

$$x=2t-7, \quad y=t-2, \quad z=t.$$

THE DISTANCE FROM A POINT TO A LINE

Consider the distance from the fixed point P to the line L.

The **shortest distance** from the point P to the line L is the **perpendicular** distance from P to the line. That is, the shortest distance is the distance PQ where the point Q lies on the line L and is such that PQ is perpendicular to the line, as shown in the diagram below.



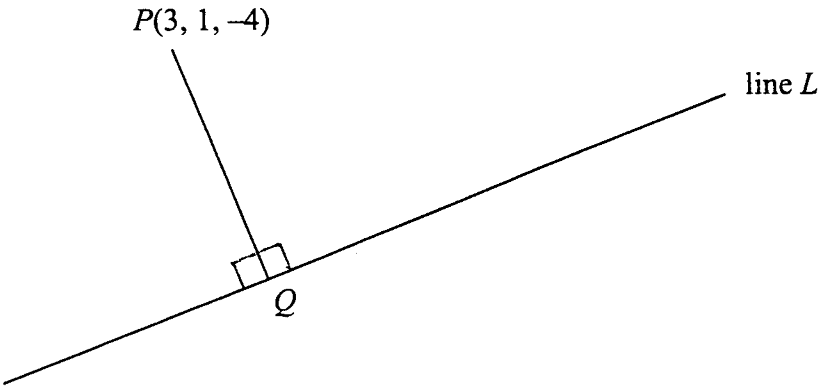
Note that the vector $\overrightarrow{PQ}$ is perpendicular to a vector in the direction of the line L.

Worked Example

Find the shortest distance from the point $P(3, 1, -4)$ to the line L with equation

$$\frac{x-1}{2} = \frac{y+10}{2} = \frac{z-10}{-3}.$$

Solution



Firstly, express the equation of line L in parametric form.

Line L: $\dfrac{x-1}{2} = \dfrac{y+10}{2} = \dfrac{z-10}{-3} = t$

$\Rightarrow \quad x = 2t+1, \quad y = 2t-10, \quad z = -3t+10$

Since Q lies on line L, the coordinates of Q will be $(2t+1,\ 2t-10,\ -3t+10)$ for some value of t.

Then $\overrightarrow{PQ} = \begin{pmatrix} 2t+1-3 \\ 2t-10-1 \\ -3t+10-(-4) \end{pmatrix} = \begin{pmatrix} 2t-2 \\ 2t-11 \\ -3t+14 \end{pmatrix}$.

Now the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$ is a vector in the direction of line L.

$\overrightarrow{PQ}$ is perpendicular to $\mathbf{a}$, hence $\overrightarrow{PQ} \bullet \mathbf{a} = 0$

$$\begin{aligned}
&\Rightarrow \quad 2(2t-2) + 2(2t-11) - 3(-3t+14) = 0 \\
&\Rightarrow \quad 4t - 4 + 4t - 22 + 9t - 42 = 0 \\
&\Rightarrow \quad 17t - 68 = 0 \\
&\Rightarrow \quad t = 4
\end{aligned}$$

So $\overrightarrow{PQ} = \begin{pmatrix} 2t+1-3 \\ 2t-10-1 \\ -3t+10-(-4) \end{pmatrix} = \begin{pmatrix} 2\times4-2 \\ 2\times4-11 \\ -3\times4+14 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 2 \end{pmatrix}$.

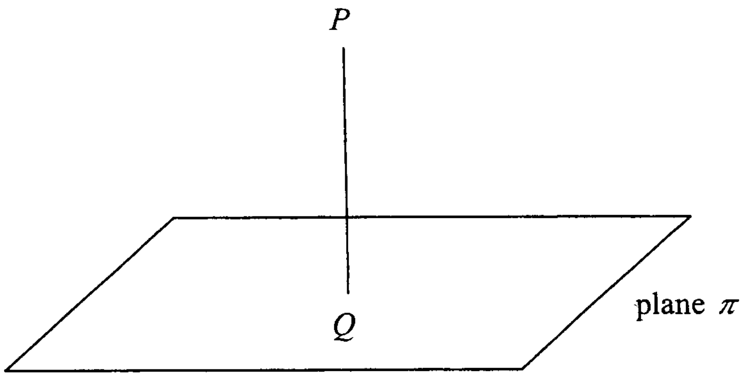
$\left|\overrightarrow{PQ}\right| = \sqrt{6^2 + (-3)^2 + 2^2} = \sqrt{49} = 7$

The shortest distance from the point P to the line L is therefore 7 units.

THE DISTANCE FROM A POINT TO A PLANE

Consider the distance from the fixed point P to the plane π.

The **shortest distance** from the point P to the plane π is the **perpendicular** distance from P to the plane. That is, the shortest distance is the distance PQ where the point Q lies in the plane π and is such that PQ is perpendicular to the plane, as shown in the diagram below.

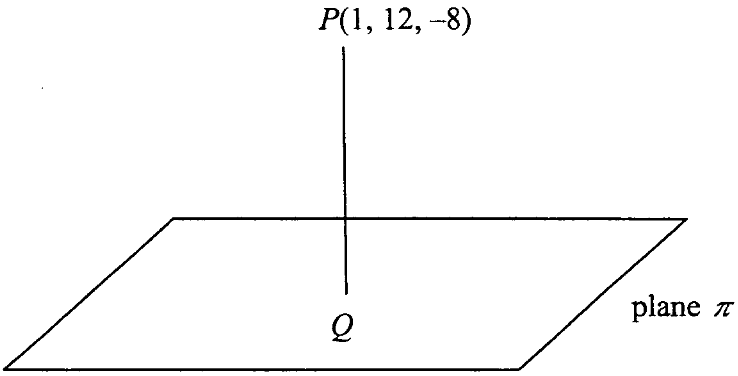


Note that the vector $\overrightarrow{PQ}$ is parallel to a vector normal to the plane π.

Worked Example

Find the shortest distance from the point $P(1, 12, -8)$ to the plane π with equation $x - 2y + 2z = 6$.

Solution



The vector $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ is normal to the plane π and is therefore a vector in the direction of the line PQ.

Equation of PQ in symmetric form:

$$\frac{x-1}{1}=\frac{y-12}{-2}=\frac{z+8}{2}=t$$

Q is the point of intersection of the line PQ and the plane π.

To find the coordinates of Q, the equation of PQ must be expressed in parametric form.

$$PQ: \quad x=t+1, \quad y=-2t+12, \quad z=2t-8$$

The line PQ intersects the plane π when

$$\begin{aligned} & x-2y+2z=6 \\ \Rightarrow \quad & t+1-2(-2t+12)+2(2t-8)=6 \\ \Rightarrow \quad & t+1+4t-24+4t-16=6 \\ \Rightarrow \quad & 9t-39=6 \\ \Rightarrow \quad & t=5 \end{aligned}$$

The coordinates of Q can now be found by substitution:

$$\begin{aligned} & x=t+1=5+1=6 \\ & y=-2t+12=-2\times 5+12=2 \\ & z=2t-8=2\times 5-8=2 \end{aligned}$$

Hence Q is the point (6, 2, 2).

Now P is (1, 12, –8) and Q is (6, 2, 2), hence $\overrightarrow{PQ}=\begin{pmatrix} 6-1 \\ 2-12 \\ 2-(-8) \end{pmatrix}=\begin{pmatrix} 5 \\ -10 \\ 10 \end{pmatrix}$.

$$\left|\overrightarrow{PQ}\right|=\sqrt{5^2+(-10)^2+10^2}=\sqrt{225}=15$$

The shortest distance from the point P to the plane π is therefore 15 units.