ADVANCED HIGHER MATHEMATICS

FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

Consider the first order differential equation

$$\frac{dy}{dx} + y = 6e^{2x}.$$

The variables cannot be separated in this differential equation and another method must be used to find the general solution.

The above differential equation is an example of a **first-order linear differential equation**.

In general, a first-order linear differential equation is of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x),}$$

where $P(x)$ and $Q(x)$ are function of x only.

To find the general solution, first find the **integrating factor**, $I(x)$, where

$$\boxed{I(x) = e^{\int P(x)dx}.}$$

It can be shown that the general solution of the differential equation is then given by

$$\boxed{I(x)y = \int I(x)Q(x)dx.}$$

Worked Example 1

Find the general solution of the differential equation

$$\frac{dy}{dx} + y = 6e^{2x}.$$

Solution

This is a first-order linear differential equation with $P(x) = 1$ and $Q(x) = 6e^{2x}$.

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int 1dx} = e^x$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\begin{aligned} \Rightarrow \quad e^x y &= \int e^x \cdot 6e^{2x} dx \\ &= \int 6e^{3x} dx \\ &= 6 \cdot \frac{1}{3} e^{3x} + C \\ &= 2e^{3x} + C \end{aligned}$$

$$\Rightarrow \quad y = 2e^{2x} + \frac{C}{e^x}$$

Worked Example 2

Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x.$$

Solution

$$\frac{dy}{dx} + \frac{y}{x} = x \quad \Rightarrow \quad \frac{dy}{dx} + \frac{1}{x}y = x$$

This is a first-order linear differential equation with $P(x) = \frac{1}{x}$ and $Q(x) = x$.

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\begin{aligned} \Rightarrow \quad xy &= \int x \cdot x dx \\ &= \int x^2 dx \\ &= \frac{1}{3}x^3 + C \end{aligned}$$

$$\Rightarrow \quad y = \frac{1}{3}x^2 + \frac{C}{x}$$

YOU CAN NOW ATTEMPT QUESTION 1 OF THE WORKSHEET "FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS."

Worked Example 3

Find the general solution of the differential equation

$$x\frac{dy}{dx} - 2y = x^3 \sin x .$$

Solution

The differential equation must be written in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$:

$$x\frac{dy}{dx} - 2y = x^3 \sin x \qquad [\div x]$$

$$\Rightarrow \qquad \frac{dy}{dx} - \frac{2y}{x} = x^2 \sin x$$

$$\Rightarrow \qquad \frac{dy}{dx} - \frac{2}{x}y = x^2 \sin x$$

This is a first-order linear differential equation with $P(x) = -\frac{2}{x}$ and $Q(x) = x^2 \sin x$.

Integrating Factor: $I(x) = e^{\int P(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}$

General Solution: $I(x)y = \int I(x)Q(x)dx$

$$\Rightarrow \qquad \frac{1}{x^2}y = \int \frac{1}{x^2} \cdot x^2 \sin x dx$$

$$= \int \sin x dx$$

$$= -\cos x + C$$

$$\Rightarrow \qquad y = x^2(C - \cos x)$$

YOU CAN NOW ATTEMPT QUESTIONS 2 TO 5 OF THE WORKSHEET "FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS."

Worked Example 4

Find the particular solution of the differential equation

$$x\frac{dy}{dx}+3y=5x^2,$$

given that $y=3$ when $x=1$.

Solution

The differential equation must be written in the standard form $\dfrac{dy}{dx}+P(x)y=Q(x)$:

$$x\frac{dy}{dx}+3y=5x^2 \qquad [\div x]$$

$$\Rightarrow \qquad \frac{dy}{dx}+\frac{3y}{x}=5x$$

$$\Rightarrow \qquad \frac{dy}{dx}+\frac{3}{x}y=5x$$

This is a first-order linear differential equation with $P(x)=\dfrac{3}{x}$ and $Q(x)=5x$.

Integrating Factor: $I(x)=e^{\int P(x)dx}=e^{\int \frac{3}{x}dx}=e^{3\ln x}=e^{\ln(x^3)}=x^3$

General Solution: $I(x)y=\int I(x)Q(x)dx$

$$\Rightarrow \qquad x^3y=\int x^3\cdot 5x dx$$
$$=\int 5x^4 dx$$
$$=5\cdot\frac{1}{5}x^5+C$$
$$=x^5+C$$

$$\Rightarrow \qquad y=x^2+\frac{C}{x^3}$$

$y=3$ when $x=1$ $\quad\Rightarrow\quad 3=1^2+\dfrac{C}{1^3}$

$\Rightarrow\quad 1+C=3$

$\Rightarrow\quad C=2$

Hence $y = x^2 + \dfrac{2}{x^3}$.

YOU CAN NOW ATTEMPT QUESTIONS 6 TO 10 OF THE WORKSHEET "FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS."