SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

We will consider second-order linear differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x),$$

where a, b and c are **constants** and $f(x)$ is a function of x only.

If $f(x)=0$, the differential equation is said to be **homogeneous**.

If $f(x)\neq 0$, the differential equation is said to be **non-homogeneous**.

HOMOGENEOUS SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

A homogeneous second-order linear differential equation is of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where a, b and c are **constants**.

The general solution of this differential equation is found as follows:

STEP 1

Form the quadratic equation

$$ak^2 + bk + c = 0.$$

This equation is known as the **auxiliary equation**.

STEP 2

Find the roots of the auxiliary equation.

STEP 3

The nature of the general solution depends on the nature of the roots of the auxiliary equation.

There are three possibilities:

(1) If the roots are **real and distinct**, say $k = p$ and $k = q$ (where $p \neq q$), then the general solution of the differential equation is of the form

$$y = Ae^{px} + Be^{qx},$$

where A and B are constants.

(2) If the roots are **real and equal**, say $k = p$ is a double root, then the general solution of the differential equation is of the form

$$y = (Ax + B)e^{px},$$

where A and B are constants.

(3) If the roots are **complex conjugates**, say $k = p \pm qi$, then the general solution of the differential equation is of the form

$$y = e^{px}(A \sin qx + B \cos qx),$$

where A and B are constants.

Worked Example 1

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0 .$$

Solution

Auxiliary Equation: $k^2 + 2k - 8 = 0$

$\Rightarrow \quad (k+4)(k-2) = 0$

$\Rightarrow \quad k = -4 \text{ or } k = 2$

The roots are real and distinct.

General Solution: $y = Ae^{-4x} + Be^{2x}$

Worked Example 2

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$$

Solution

Auxiliary Equation: $k^2 - 6k + 9 = 0$

$\Rightarrow$ $(k-3)(k-3) = 0$

$\Rightarrow$ $k = 3$ (double root)

The roots are real and equal.

General Solution: $y = (Ax + B)e^{3x}$

Worked Example 3

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0.$$

Solution

Auxiliary Equation: $k^2 + 4k + 13 = 0$

The quadratic does not factorise, so we must solve using the quadratic formula.

$a = 1,\ b = 4,\ c = 13$

$b^2 - 4ac = 4^2 - 4 \times 1 \times 13 = -36$

$$\begin{aligned} k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{-36}}{2} \\ &= \frac{-4 \pm 6i}{2} \\ &= -2 \pm 3i \end{aligned}$$

The roots are complex conjugates.

General Solution: $y = e^{-2x}(A\sin 3x + B\cos 3x)$

YOU CAN NOW ATTEMPT QUESTION 1 OF THE WORKSHEET "SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS (HOMOGENEOUS)."

Worked Example 4

Find the particular solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0,$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 5$.

Solution

Auxiliary Equation: $\quad k^2 - 4k + 3 = 0$

$\Rightarrow \quad (k-3)(k-1) = 0$

$\Rightarrow \quad k = 3$ or $k = 1$

The roots are real and distinct.

General Solution: $\quad y = Ae^{3x} + Be^{x}$

When $x = 0$, $y = 1$ $\Rightarrow \quad 1 = Ae^{0} + Be^{0}$

$\Rightarrow \quad A + B = 1 \quad$ ...(1)

$y = Ae^{3x} + Be^{x} \quad \Rightarrow \quad \frac{dy}{dx} = 3Ae^{3x} + Be^{x}$

When $x = 0$, $\frac{dy}{dx} = 5$ $\Rightarrow \quad 5 = 3Ae^{0} + Be^{0}$

$\Rightarrow \quad 3A + B = 5 \quad$ ...(2)

(2) – (1) $\Rightarrow \quad 2A = 4$

$\Rightarrow \quad A = 2$

Sub. in (1) $\Rightarrow \quad A + B = 1$

$\Rightarrow \quad 2 + B = 1$

$\Rightarrow \quad B = -1$

Hence $y = 2e^{3x} - e^{x}$.

Worked Example 5

Find the particular solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0,$$

given that when $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$.

Solution

Auxiliary Equation: $k^2 + 4k + 4 = 0$

$\Rightarrow \quad (k+2)(k+2) = 0$

$\Rightarrow \quad k = -2$ (double root)

The roots are real and equal.

General Solution: $y = (Ax + B)e^{-2x}$

When $x = 0$, $y = 1$ $\Rightarrow \quad 1 = (B)e^0$

$\Rightarrow \quad B = 1$

$y = (Ax + B)e^{-2x} \quad \Rightarrow \quad \frac{dy}{dx} = (Ax + B)\cdot(-2e^{-2x}) + e^{-2x}\cdot A$ [using the product rule]

$= -2e^{-2x}(Ax + B) + Ae^{-2x}$

When $x = 0$, $\frac{dy}{dx} = 2$ $\Rightarrow \quad 2 = -2e^0(B) + Ae^0$

$\Rightarrow \quad -2B + A = 2$

$\Rightarrow \quad -2(1) + A = 2$

$\Rightarrow \quad A = 4$

Hence $y = (4x + 1)e^{-2x}$.

Worked Example 6

Find the particular solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0,$$

given that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 1$.

Solution

Auxiliary Equation: $k^2 - 4k + 5 = 0$

The quadratic does not factorise, so we must solve using the quadratic formula.

$a = 1$, $b = -4$, $c = 5$

$b^2 - 4ac = (-4)^2 - 4 \times 1 \times 5 = -4$

$$\begin{aligned} k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4 \pm 2i}{2} \\ &= 2 \pm i \end{aligned}$$

The roots are complex conjugates.

General Solution: $y = e^{2x}(A\sin x + B\cos x)$

When $x = 0$, $y = 2$ $\Rightarrow$ $2 = e^0(A\sin 0 + B\cos 0)$

$\Rightarrow$ $B = 2$

$$y = e^{2x}(A\sin x + B\cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{2x}(A\cos x - B\sin x) + (A\sin x + B\cos x)\cdot 2e^{2x} \quad \text{[using the product rule]}$$

$$= e^{2x}(A\cos x - B\sin x) + 2e^{2x}(A\sin x + B\cos x)$$

When $x = 0$, $\frac{dy}{dx} = 1$ $\Rightarrow$ $1 = e^0(A\cos 0 - B\sin 0) + 2e^0(A\sin 0 + B\cos 0)$

$\Rightarrow$ $A + 2B = 1$

$$\Rightarrow \quad A + 2(2) = 1$$
$$\Rightarrow \quad A = -3$$

Hence $y = e^{2x}(-3\sin x + 2\cos x)$, i.e. $y = e^{2x}(2\cos x - 3\sin x)$.

YOU CAN NOW ATTEMPT QUESTION 2 OF THE WORKSHEET "SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS (HOMOGENEOUS)."

NON-HOMOGENEOUS SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

A **non-homogeneous second-order linear differential equation** is of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where a, b and c are constants and $f(x) \neq 0$.

The general solution of this differential equation is found as follows:

STEP 1

Find the general solution of the **homogeneous** differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

as before.

The general solution of this homogeneous differential equation is known as the **complementary function.**

STEP 2

Find a particular solution of the non-homogeneous differential equation. This solution is known as the **particular integral.**

The form of the particular integral will be similar to the form of $f(x)$. Advice on finding the particular integral will be given later.

STEP 3

The general solution of the non-homogeneous differential equation is the sum of the complementary function and the particular integral.

ADVICE ON FINDING THE PARTICULAR INTEGRAL

The form of the particular integral will be similar to the form of $f(x)$.

Example 1

If $f(x) = 2x + 1$, the particular integral will be of the form $y = px + q$, where p and q are constants to be determined.

Example 2

If $f(x) = x^2 - 1$, the particular integral will be of the form $y = px^2 + qx + r$, where p, q and r are constants to be determined. Note that the term qx must be included in the particular integral as $f(x) = x^2 + 0x - 1$.

Example 3

If $f(x) = 4e^{2x}$, the particular integral will be of the form $y = pe^{2x}$, where p is a constant to be determined.

Example 4

If $f(x) = 2\sin x + \cos x$, the particular integral will be of the form $y = p\sin x + q\cos x$, where p and q are constants to be determined.

Example 5

If $f(x) = 3\sin 2x$, the particular integral will be of the form $y = p\sin 2x + q\cos 2x$, where p and q are constants to be determined. Note that the term $q\cos 2x$ must be included in the particular integral as $f(x) = 3\sin 2x + 0\cos 2x$.

Worked Example 1

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x + 1.$$

Solution

Complementary Function: $k^2 - k - 2 = 0$

$\Rightarrow \quad (k-2)(k+1) = 0$

$\Rightarrow \quad k = 2$ or $k = -1$

The roots are real and distinct $\Rightarrow \quad y = Ae^{2x} + Be^{-x}$

Particular Integral: Try $y = px + q$.

Then $\dfrac{dy}{dx} = p$ and $\dfrac{d^2y}{dx^2} = 0$.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 4x + 1$$

$\Rightarrow \quad 0 - p - 2(px + q) = 4x + 1$

$\Rightarrow \quad -p - 2px - 2q = 4x + 1$

Equating coefficients of x $\Rightarrow \quad -2p = 4$

$\Rightarrow \quad p = -2$

Equating constants $\Rightarrow \quad -p - 2q = 1$

$\Rightarrow \quad 2 - 2q = 1$

$\Rightarrow \quad 2q = 1$

$\Rightarrow \quad q = \dfrac{1}{2}$

$y = px + q \quad \Rightarrow \quad y = -2x + \dfrac{1}{2}$

General Solution: $y = Ae^{2x} + Be^{-x} - 2x + \dfrac{1}{2}$

Worked Example 2

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 1.$$

Solution

Complementary Function: $k^2 - 4k + 4 = 0$

$\Rightarrow \quad (k-2)(k-2) = 0$

$\Rightarrow \quad k = 2$ (double root)

The roots are real and equal $\quad \Rightarrow \quad y = (Ax + B)e^{2x}$

Particular Integral: Try $y = px^2 + qx + r$.

Then $\dfrac{dy}{dx} = 2px + q$ and $\dfrac{d^2y}{dx^2} = 2p$.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2x^2 + 1$$

$\Rightarrow \quad 2p - 4(2px + q) + 4(px^2 + qx + r) = 2x^2 + 1$

$\Rightarrow \quad 2p - 8px - 4q + 4px^2 + 4qx + 4r = 2x^2 + 1$

Equating coefficients of x^2 $\quad \Rightarrow \quad 4p = 2$

$\Rightarrow \quad p = \dfrac{1}{2}$

Equating coefficients of x $\quad \Rightarrow \quad -8p + 4q = 0$

$\Rightarrow \quad -8\left(\dfrac{1}{2}\right) + 4q = 0$

$\Rightarrow \quad -4 + 4q = 0$

$\Rightarrow \quad q = 1$

Equating constants $\quad \Rightarrow \quad 2p - 4q + 4r = 1$

$\Rightarrow \quad 2\left(\dfrac{1}{2}\right) - 4(1) + 4r = 1$

$\Rightarrow \quad -3 + 4r = 1$

$\Rightarrow \quad r = 1$

$y = px^2 + qx + r \quad \Rightarrow \quad y = \dfrac{1}{2}x^2 + x + 1$

General Solution: $y = (Ax + B)e^{2x} + \frac{1}{2}x^2 + x + 1$

YOU CAN NOW ATTEMPT QUESTION 1 OF THE WORKSHEET "NON-HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS."

Worked Example 3

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}.$$

Solution

Complementary Function: $k^2 + 2k - 3 = 0$

$\Rightarrow \quad (k+3)(k-1) = 0$

$\Rightarrow \quad k = -3$ or $k = 1$

The roots are real and distinct $\Rightarrow \quad y = Ae^{-3x} + Be^{x}$

Particular Integral: Try $y = pe^{2x}$.

Then $\frac{dy}{dx} = 2pe^{2x}$ and $\frac{d^2y}{dx^2} = 4pe^{2x}$.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 10e^{2x}$$

$\Rightarrow \quad 4pe^{2x} + 2(2pe^{2x}) - 3(pe^{2x}) = 10e^{2x}$

$\Rightarrow \quad 4pe^{2x} + 4pe^{2x} - 3pe^{2x} = 10e^{2x}$

$\Rightarrow \quad 5pe^{2x} = 10e^{2x}$

Equating coefficients of e^{2x} $\Rightarrow \quad 5p = 10$

$\Rightarrow \quad p = 2$

$y = pe^{2x} \quad \Rightarrow \quad y = 2e^{2x}$

General Solution: $y = Ae^{-3x} + Be^{x} + 2e^{2x}$

Worked Example 4

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} + 16y = e^{-2x}.$$

Solution

Complementary Function: $k^2 + 16 = 0$

$\Rightarrow \quad k^2 = -16$

$\Rightarrow \quad k = \pm 4i \ (= 0 \pm 4i)$

The roots are complex conjugates $\Rightarrow \quad y = e^{0x}(A\sin 4x + B\cos 4x)$

$\Rightarrow \quad y = A\sin 4x + B\cos 4x$

Particular Integral: Try $y = pe^{-2x}$.

Then $\dfrac{dy}{dx} = -2pe^{-2x}$ and $\dfrac{d^2y}{dx^2} = 4pe^{-2x}$.

$$\frac{d^2y}{dx^2} + 16y = e^{-2x}$$

$\Rightarrow \quad 4pe^{-2x} + 16(pe^{-2x}) = e^{-2x}$

$\Rightarrow \quad 20pe^{-2x} = e^{-2x}$

Equating coefficients of e^{-2x} $\Rightarrow \quad 20p = 1$

$\Rightarrow \quad p = \dfrac{1}{20}$

$y = pe^{-2x} \quad \Rightarrow \quad y = \dfrac{1}{20}e^{-2x}$

General Solution: $y = A\sin 4x + B\cos 4x + \dfrac{1}{20}e^{-2x}$

YOU CAN NOW ATTEMPT QUESTION 2 OF THE WORKSHEET "NON-HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS."

Worked Example 5

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10\sin x.$$

Solution

Complementary Function: $k^2 - 4k + 3 = 0$

$\Rightarrow \quad (k-3)(k-1) = 0$

$\Rightarrow \quad k = 3 \text{ or } k = 1$

The roots are real and distinct $\quad \Rightarrow \quad y = Ae^{3x} + Be^{x}$

Particular Integral: Try $y = p\sin x + q\cos x$.

$$\frac{dy}{dx} = p\cos x - q\sin x$$

$$\frac{d^2y}{dx^2} = -p\sin x - q\cos x$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10\sin x$$

$\Rightarrow \quad -p\sin x - q\cos x - 4(p\cos x - q\sin x) + 3(p\sin x + q\cos x) = 10\sin x$

$\Rightarrow \quad -p\sin x - q\cos x - 4p\cos x + 4q\sin x + 3p\sin x + 3q\cos x = 10\sin x$

Equating coefficients of $\sin x \Rightarrow \quad -p + 4q + 3p = 10$

$\Rightarrow \quad 2p + 4q = 10 \quad ...(1)$

Equating coefficients of $\cos x \Rightarrow \quad -q - 4p + 3q = 0$

$\Rightarrow \quad -4p + 2q = 0 \quad ...(2)$

Solving equations (1) and (2) simultaneously gives $p = 1$ and $q = 2$.

$y = p\sin x + q\cos x \quad \Rightarrow \quad y = \sin x + 2\cos x$

General Solution: $\quad y = Ae^{3x} + Be^{x} + \sin x + 2\cos x$

YOU CAN NOW ATTEMPT QUESTION 3 OF THE WORKSHEET "NON-HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS."

Worked Example 6 **(Exam Question)**

Find the particular solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x,$$

given that when $x = 0$, $y = 2$ and $\frac{dy}{dx} = 1$.

Solution

Complementary Function: $k^2 - 4k + 4 = 0$

$\Rightarrow \quad (k-2)(k-2) = 0$

$\Rightarrow \quad k = 2$ (double root)

The roots are real and equal $\Rightarrow \quad y = (Ax + B)e^{2x}$

Particular Integral: Try $y = pe^x$.

Then $\frac{dy}{dx} = pe^x$ and $\frac{d^2y}{dx^2} = pe^x$.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

$\Rightarrow \quad pe^x - 4(pe^x) + 4(pe^x) = e^x$

$\Rightarrow \quad pe^x = e^x$

Equating coefficients of e^x $\Rightarrow \quad p = 1$

$y = pe^x \quad \Rightarrow \quad y = e^x$

General Solution: $y = (Ax + B)e^{2x} + e^x$

When $x = 0$, $y = 2$ $\Rightarrow \quad 2 = (B)e^0 + e^0$

$\Rightarrow \quad B + 1 = 2$

$\Rightarrow \quad B = 1$

$$y = (Ax + B)e^{2x} + e^x$$

$\Rightarrow \quad \frac{dy}{dx} = (Ax + B) \cdot 2e^{2x} + e^{2x} \cdot A + e^x$ [using the product rule to differentiate $(Ax + B)e^{2x}$]

$$= 2e^{2x}(Ax + B) + Ae^{2x} + e^x$$

When $x = 0$, $\frac{dy}{dx} = 1 \quad \Rightarrow \quad 1 = 2e^0(B) + Ae^0 + e^0$

$$\Rightarrow \quad 2B + A + 1 = 1$$

$$\Rightarrow \quad 2(1) + A + 1 = 1$$

$$\Rightarrow \quad A = -2$$

Hence $y = (-2x + 1)e^{2x} + e^x$,

i.e. $y = (1 - 2x)e^{2x} + e^x$.

AMENDED PARTICULAR INTEGRALS

Occasionally, the form of the particular integral must be amended. The following example illustrates this.

Worked Example

Find the general solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^{3x}.$$

Solution

Complementary Function: $k^2 - 5k + 6 = 0$

$\Rightarrow \quad (k-3)(k-2) = 0$

$\Rightarrow \quad k = 3 \text{ or } k = 2$

The roots are real and distinct $\Rightarrow \quad y = Ae^{3x} + Be^{2x}$

Particular Integral: Try $y = pe^{3x}$.

Then $\dfrac{dy}{dx} = 3pe^{3x}$ and $\dfrac{d^2y}{dx^2} = 9pe^{3x}$.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^{3x}$$

$\Rightarrow \quad 9pe^{3x} - 5(3pe^{3x}) + 6(pe^{3x}) = 2e^{3x}$

$\Rightarrow \quad 9pe^{3x} - 15pe^{2x} + 6pe^{3x} = 2e^{3x}$

$\Rightarrow \quad 0 = 2e^{3x}$???

Now try the amended particular integral $y = pxe^{3x}$.

$\dfrac{dy}{dx} = px \cdot 3e^{3x} + e^{3x} \cdot p$ [using the product rule]

$= (3px + p)e^{3x}$

$\dfrac{d^2y}{dx^2} = (3px + p) \cdot 3e^{3x} + e^{3x} \cdot 3p$ [using the product rule]

$= (9px + 3p)e^{3x} + 3pe^{3x}$

$= (9px + 3p + 3p)e^{3x}$

$= (9px + 6p)e^{3x}$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^{3x}$$

$\Rightarrow \quad (9px + 6p)e^{3x} - 5(3px + p)e^{3x} + 6pxe^{3x} = 2e^{3x}$

$\Rightarrow \quad (9px + 6p - 15px - 5p + 6px)e^{3x} = 2e^{3x}$

$\Rightarrow \quad pe^{3x} = 2e^{3x}$

Equating coefficients of e^{3x} $\Rightarrow$ $p = 2$

$y = pxe^{3x}$ $\Rightarrow$ $y = 2xe^{3x}$

General Solution: $y = Ae^{3x} + Be^{2x} + 2xe^{3x}$,
i.e. $y = (2x + A)e^{3x} + Be^{2x}$

[Note that the particular integral $y = pe^{2x}$ fails here since a term of this form (Ae^{3x}) is already contained in the complementary function.]

YOU CAN NOW ATTEMPT QUESTION 4 OF THE WORKSHEET "NON-HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS."