ADVANCED HIGHER MATHEMATICS

PROOF BY INDUCTION

Recall that a mathematical **statement** is either **true** or **false**.

Let p_n be a given statement about the positive integer n.

Example

Let p_n be the statement that $n^2 + 2n$ is divisible by 3.

p_1 is the statement that $1^2 + 2 \times 1$ is divisible by 3.
Since $1^2 + 2 \times 1 = 3$, the statement p_1 is true.

p_2 is the statement that $2^2 + 2 \times 2$ is divisible by 3.
Since $2^2 + 2 \times 2 = 8$, the statement p_2 is false.

Hence the statement p_n is **not** true for all positive integers n.

The method of **proof by induction** can be used to prove that a given statement p_n is true for **all** positive integers n.

STEP 1

Prove that the statement p_1 is true.
This is known as the **basis** for the proof by induction.

STEP 2

Assume that the statement p_n is true for some positive integer n.

STEP 3

Use the assumption made in STEP 2 to **prove** that the statement p_{n+1} is then also true.
This is known as the **inductive step**.

STEP 4

This completes the proof by induction that the statement p_n is true for **all** positive integers n.

The logic behind the method of proof by induction can be explained as follows.

By following the steps outlined, we have shown that:

(1) The statement p_1 is true;

(2) If the statement p_n is true, then the statement p_{n+1} is also true for all positive integers n. This can be written as

$$p_n \text{ true} \quad \Rightarrow \quad p_{n+1} \text{ true} \quad \text{for all positive integers } n.$$

We have proved that the statement p_1 is true, therefore by (2) above, the statement p_2 is also true. This gives rise to the following inductive chain using (2) above:

$$\begin{aligned} p_1 \text{ true} &\quad \Rightarrow \quad p_2 \text{ true;} \\ p_2 \text{ true} &\quad \Rightarrow \quad p_3 \text{ true;} \\ p_3 \text{ true} &\quad \Rightarrow \quad p_4 \text{ true;} \end{aligned}$$

and so on.

Thus the statement p_n is true for all positive integers n.

Note that we **must** prove that the statement p_1 is true to begin this chain of induction.

NOTE

The method of proof by induction can also be used to prove that a given statement p_n is true for all integers $n \geq m$ as follows:

(1) Prove that the statement p_m is true.

(2) Assuming that the statement p_n is true for some integer $n \geq m$, prove that the statement p_{n+1} is also true.

When attempting to prove the inductive step that p_{n+1} is true, it is useful to note that

$$\sum_{k=1}^{n+1} f(k) = \sum_{k=1}^{n} f(k) + f(n+1).$$

Worked Example 1

Prove by induction that $\sum_{k=1}^{n}(2k-1)=n^2$ for all positive integers n.

Solution

Let p_n be the statement that $\sum_{k=1}^{n}(2k-1)=n^2$.

BASIS

p_1 is the statement that $\sum_{k=1}^{1}(2k-1)=1^2$.

$\text{LHS}=\sum_{k=1}^{1}(2k-1)=2\times 1-1=1$

$\text{RHS}=1^2=1$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,

i.e. assume that $\sum_{k=1}^{n}(2k-1)=n^2$.

AIM

Prove that the statement p_{n+1} is also true,

i.e. prove that $\sum_{k=1}^{n+1}(2k-1)=(n+1)^2$.

METHOD

$$\begin{aligned}\sum_{k=1}^{n+1}(2k-1)&=\sum_{k=1}^{n}(2k-1)+\{2(n+1)-1\}\\&=n^2+2(n+1)-1 \quad \text{[since we have assumed that } p_n \text{ is true]}\\&=n^2+2n+2-1\\&=n^2+2n+1\\&=(n+1)(n+1)\\&=(n+1)^2\end{aligned}$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

Worked Example 2

Prove by induction that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Solution

Let p_n be the statement that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

BASIS

p_1 is the statement that $\sum_{k=1}^{1} k^2 = \frac{1}{6}(1)(1+1)(2(1)+1)$.

$$\text{LHS} = \sum_{k=1}^{1} k^2 = 1^2 = 1$$

$$\text{RHS} = \frac{1}{6}(1)(1+1)(2(1)+1) = \frac{1}{6}(1)(2)(3) = 1$$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,

i.e. assume that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

AIM

Prove that the statement p_{n+1} is also true,

i.e. prove that
$$\sum_{k=1}^{n+1} k^2 = \frac{1}{6}(n+1)\{(n+1)+1\}\{2(n+1)+1\}$$
$$= \frac{1}{6}(n+1)(n+2)(2n+3)$$

METHOD

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2$$

$$= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{[since we have assumed that } p_n \text{ is true]}$$

$$= \frac{1}{6}(n+1)\{n(2n+1) + 6(n+1)\}$$

$$= \frac{1}{6}(n+1)(2n^2 + n + 6n + 6)$$

$$= \frac{1}{6}(n+1)(2n^2 + 7n + 6)$$

$$= \frac{1}{6}(n+1)(2n+3)(n+2)$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3)$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

Worked Example 3

(a) Prove by induction that $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$ for all positive integers n.

(b) Hence evaluate $\sum_{k=11}^{20} k(k+1)$.

Solution

(a) Let p_n be the statement that $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$.

BASIS

p_1 is the statement that $\sum_{k=1}^{1} k(k+1) = \frac{1}{3}(1)(1+1)(1+2)$.

$$\text{LHS} = \sum_{k=1}^{1} k(k+1) = (1)(1+1) = 2$$

$$\text{RHS} = \frac{1}{3}(1)(1+1)(1+2) = \frac{1}{3}(1)(2)(3) = 2$$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,

i.e. assume that $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$.

AIM

Prove that the statement p_{n+1} is also true,

i.e. prove that
$$\sum_{k=1}^{n+1} k(k+1) = \frac{1}{3}(n+1)\{(n+1)+1\}\{(n+1)+2\}$$
$$= \frac{1}{3}(n+1)(n+2)(n+3)$$

METHOD

$$\sum_{k=1}^{n+1} k(k+1) = \sum_{k=1}^{n} k(k+1) + (n+1)\{(n+1)+1\}$$

$$= \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2)$$ [since we have assumed that p_n is true]

$$= \frac{1}{3}(n+1)\{n(n+2) + 3(n+2)\}$$

$$= \frac{1}{3}(n+1)(n^2 + 2n + 3n + 6)$$

$$= \frac{1}{3}(n+1)(n^2 + 5n + 6)$$

$$= \frac{1}{3}(n+1)(n+2)(n+3)$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

(b) $$\sum_{k=11}^{20} k(k+1) = \sum_{k=1}^{20} k(k+1) - \sum_{k=1}^{10} k(k+1)$$

Now $\sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2)$.

$$\sum_{k=1}^{20} k(k+1) = \frac{1}{3}(20)(20+1)(20+2) = \frac{1}{3}(20)(21)(22) = 3080$$

$$\sum_{k=1}^{10} k(k+1) = \frac{1}{3}(10)(10+1)(10+2) = \frac{1}{3}(10)(11)(12) = 440$$

Hence $\sum_{k=11}^{20} k(k+1) = 3080 - 440 = 2640$.

Worked Example 4

Prove by induction that

$$2+5+8+....+(3n-1)=\frac{1}{2}n(3n+1)$$

for all positive integers n.

Solution

Let p_n be the statement that $2+5+8+....+(3n-1)=\frac{1}{2}n(3n+1)$,

i.e. that $\sum_{k=1}^{n}(3k-1)=\frac{1}{2}n(3n+1)$.

BASIS

p_1 is the statement that $\sum_{k=1}^{1}(3k-1)=\frac{1}{2}(1)(3(1)+1)$.

$$\text{LHS}=\sum_{k=1}^{1}(3k-1)=3\times 1-1=2$$

$$\text{RHS}=\frac{1}{2}(1)(3(1)+1)=\frac{1}{2}(1)(4)=2$$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,

i.e. assume that $\sum_{k=1}^{n}(3k-1)=\frac{1}{2}n(3n+1)$.

AIM

Prove that the statement p_{n+1} is also true,

i.e. prove that
$$\sum_{k=1}^{n+1}(3k-1)=\frac{1}{2}(n+1)\{3(n+1)+1\}$$
$$=\frac{1}{2}(n+1)(3n+4)$$

METHOD

$$\sum_{k=1}^{n+1}(3k-1)=\sum_{k=1}^{n}(3k-1)+\{3(n+1)-1\}$$

$$=\frac{1}{2}n(3n+1)+3(n+1)-1 \qquad \text{[since we have assumed that } p_n \text{ is true]}$$

$$=\frac{1}{2}n(3n+1)+3n+2$$

$$=\frac{1}{2}\{n(3n+1)+6n+4\}$$

$$=\frac{1}{2}(3n^2+n+6n+4)$$

$$=\frac{1}{2}(3n^2+7n+4)$$

$$=\frac{1}{2}(3n+4)(n+1)$$

$$=\frac{1}{2}(n+1)(3n+4)$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

[*Note that it is convenient to express the statement* p_n *using* $\sum$ *notation.*]

Worked Example 5

Prove by induction that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for all positive integers n.

Solution

Let p_n be the statement that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

BASIS

p_1 is the statement that $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1+1}$.

$$\text{LHS} = \sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$\text{RHS} = \frac{1}{1+1} = \frac{1}{2}$$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,

i.e. assume that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

AIM

Prove that the statement p_{n+1} is also true,

i.e. prove that $\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$.

METHOD

$$\sum_{k=1}^{n+1}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\frac{1}{k(k+1)}+\frac{1}{(n+1)\{(n+1)+1\}}$$

$$=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)} \quad \text{[since we have assumed that } p_n \text{ is true]}$$

$$=\frac{n(n+2)}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)}$$

$$=\frac{n(n+2)+1}{(n+1)(n+2)}$$

$$=\frac{n^2+2n+1}{(n+1)(n+2)}$$

$$=\frac{(n+1)(n+1)}{(n+1)(n+2)}$$

$$=\frac{n+1}{n+2}$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

[The result $\sum_{k=1}^{n}\frac{1}{k(k+1)}=\frac{n}{n+1}$ can also be proved directly by expressing $\frac{1}{k(k+1)}$ in partial fractions.

It can be shown that $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$.

Hence $\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

$$=\frac{1}{1}-\frac{1}{2} \quad (k=1)$$

$$+\frac{1}{2}-\frac{1}{3} \quad (k=2)$$

$$+\frac{1}{3}-\frac{1}{4} \quad (k=3)$$

$$+\frac{1}{4}-\frac{1}{5} \quad (k=4)$$

$$+\ldots\ldots$$

$$+\frac{1}{n}-\frac{1}{n+1} \quad (k=n)$$

$$= 1 - \frac{1}{n+1}$$ [note that the terms cancel in diagonal pairs as shown]

$$= \frac{n+1}{n+1} - \frac{1}{n+1}$$

$$= \frac{n+1-1}{n+1}$$

$$= \frac{n}{n+1}]$$

Worked Example 6

Prove by induction that $4^n - 1$ is divisible by 3 for all positive integers n.

Solution

Let p_n be the statement that $4^n - 1$ is divisible by 3.

BASIS

p_1 is the statement that $4^1 - 1$ is divisible by 3.

Now $4^1 - 1 = 3$, which is divisible by 3, hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,
i.e. assume that $4^n - 1$ is divisible by 3.

AIM

Prove that the statement p_{n+1} is also true,
i.e. prove that $4^{n+1} - 1$ is divisible by 3.

METHOD

$$\begin{aligned} 4^{n+1} - 1 &= 4 \cdot 4^n - 1 \\ &= 4\{(4^n - 1) + 1\} - 1 \\ &= 4(4^n - 1) + 4 - 1 \\ &= 4(4^n - 1) + 3 \end{aligned}$$

Now $4^n - 1$ is divisible by 3 (since we have assumed that p_n is true) and 3 is divisible by 3, hence $4^{n+1} - 1$ is divisible by 3.

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

[The inductive step can also be proved as follows:

$$\begin{aligned} 4^{n+1} - 1 &= 4 \cdot 4^n - 1 \\ &= (3+1)4^n - 1 \\ &= 3 \cdot 4^n + 4^n - 1 \\ &= 3 \cdot 4^n + (4^n - 1) \end{aligned}$$

Now $3 \cdot 4^n$ is divisible by 3 and $4^n - 1$ is divisible by 3 (since we have assumed that p_n is true), hence $4^{n+1} - 1$ is divisible by 3.]

Worked Example 7

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

Prove by induction that $A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$ for all positive integers n.

Solution

Let p_n be the statement that $A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$.

BASIS

p_1 is the statement that $A^1 = \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix}$.

$\text{LHS} = A^1 = A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

$\text{RHS} = \begin{pmatrix} 1+1 & 1 \\ -1 & 1-1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,
i.e. assume that $A^n = \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$.

AIM

Prove that the statement p_{n+1} is also true,
i.e. prove that $A^{n+1} = \begin{pmatrix} (n+1)+1 & n+1 \\ -(n+1) & 1-(n+1) \end{pmatrix} = \begin{pmatrix} n+2 & n+1 \\ -n-1 & -n \end{pmatrix}$.

METHOD

$$A^{n+1} = AA^n = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix} \quad \text{[since we have assumed that } p_n \text{ is true]}$$

$$= \begin{pmatrix} 2(n+1)+1(-n) & 2n+1(1-n) \\ -1(n+1)+0(-n) & -1n+0(1-n) \end{pmatrix}$$

$$= \begin{pmatrix} 2n+2-n & 2n+1-n \\ -n-1 & -n \end{pmatrix}$$

$$= \begin{pmatrix} n+2 & n+1 \\ -n-1 & -n \end{pmatrix}$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

Worked Example 7

Let $z = \cos\theta + i\sin\theta$.

Prove by induction that $z^n = \cos n\theta + i\sin n\theta$ for all positive integers n.

[Recall that this is de Moivre's theorem.]

Solution

Let p_n be the statement that $z^n = \cos n\theta + i\sin n\theta$.

BASIS

p_1 is the statement that $z^1 = \cos 1\theta + i\sin 1\theta$.

$\text{LHS} = z^1 = z = \cos\theta + i\sin\theta$

$\text{RHS} = \cos 1\theta + i\sin 1\theta = \cos\theta + i\sin\theta$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,
i.e. assume that $z^n = \cos n\theta + i\sin n\theta$.

AIM

Prove that the statement p_{n+1} is also true,
i.e. prove that $z^{n+1} = \cos(n+1)\theta + i\sin(n+1)\theta$.

METHOD

$z^{n+1} = zz^n = (\cos\theta + i\sin\theta)(\cos n\theta + i\sin n\theta)$ [since we have assumed that p_n is true]

$$\begin{aligned}
&= \cos\theta\cos n\theta + i\sin\theta\cos n\theta + i\cos\theta\sin n\theta + i^2\sin\theta\sin n\theta \\
&= \cos\theta\cos n\theta - \sin\theta\sin n\theta + i(\sin\theta\cos n\theta + \cos\theta\sin n\theta) \\
&= \cos(\theta + n\theta) + i\sin(\theta + n\theta) \\
&= \cos(n+1)\theta + i\sin(n+1)\theta
\end{aligned}$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.

Worked Example 8

Prove by induction that $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$ for all positive integers n.

Solution

Let p_n be the statement that $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$.

BASIS

p_1 is the statement that $\frac{d^1}{dx^1}(xe^x) = (x+1)e^x$.

$$\begin{aligned}\text{LHS} = \frac{d^1}{dx^1}(xe^x) &= \frac{d}{dx}(xe^x) \\ &= x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x) \qquad \text{[using the product rule]} \\ &= x \cdot e^x + e^x \cdot 1 \\ &= (x+1)e^x \\ &= \text{RHS}\end{aligned}$$

Hence the statement p_1 is true.

ASSUMPTION

Assume that the statement p_n is true for some positive integer n,
i.e. assume that $\frac{d^n}{dx^n}(xe^x) = (x+n)e^x$.

AIM

Prove that the statement p_{n+1} is also true,
i.e. prove that $\frac{d^{n+1}}{dx^{n+1}}(xe^x) = (x+n+1)e^x$.

METHOD

$$\frac{d^{n+1}}{dx^{n+1}}(xe^x) = \frac{d}{dx}\left\{\frac{d^n}{dx^n}(xe^x)\right\}$$

$$= \frac{d}{dx}\left\{(x+n)e^x\right\} \quad \text{[since we have assumed that } p_n \text{ is true]}$$

$$= (x+n)\cdot\frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x+n) \quad \text{[using the product rule]}$$

$$= (x+n)\cdot e^x + e^x \cdot 1$$

$$= (x+n+1)e^x$$

Hence the statement p_{n+1} is also true.

CONCLUSION

This completes the proof by induction that the statement p_n is true for all positive integers n.