Prelim Examination 2014 / 2015
(Assessing Units 1 & 2)

MATHEMATICS

Advanced Higher Grade

Time allowed - 3 hours

Read Carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions
3. **Full credit will only be given where the solution contains appropriate working**

Answer all the questions.

1. (a) Given $f(x) = x^3\tan(2x)$, where $0 < x < \frac{\pi}{4}$, obtain $f'(x)$. **3**

(b) For $y = \dfrac{1+x^2}{1+x}$, where $x \neq -1$, determine $\frac{dy}{dx}$ in simplified form. **3**

2. Find $\displaystyle\int \frac{12x^3 - 6x}{x^4 - x^2 + 1}\,dx$ **3**

3. Identify the locus in the complex plane given by $|z + i| = 2$. **3**

4. The system of equations

$$\begin{aligned} -2x + 3y + z &= 7 \\ x - 2y - 2z &= -3 \\ 2x - 2y + t^2 z &= 10 \end{aligned}$$

has no solution.

Use Gaussian elimination to find the value(s) of t. **5**

5. Determine whether the function $f(x) = x^4\sin 2x$ is odd, even or neither. Justify your answer. **3**

6. Write down the general term of the binomial expansion of $\left(2x^2 - \dfrac{4}{y^2}\right)^6$.

Hence use your expression to find the coefficient of $\dfrac{x^4}{y^8}$. **5**

7. Use integration by parts to obtain $\int 8x^2 \sin 4x \, dx$. **5**

8. A curve is defined parametrically, for all t, by the equations

$$x = 2t + \frac{1}{2}t^2, \qquad y = \frac{1}{3}t^3 - 3t.$$

Obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t. **5**

Find the values of t at which the curve has stationary points and determine their nature. **3**

Show that the curve has exactly two points of inflexion. **2**

9. A curve is defined implicitly by the equation $e^{xy} = e^{4x} - e^{5y}$.

Find an equation, in terms of x and y, for $\frac{dy}{dx}$. **4**

Hence find the gradient of the tangent to the curve at the point (1, 0). **2**

10. Express $z = (1+3i)(2-i) - \frac{2-2i}{1+i}$ in the form $a+bi$ where a and b are real numbers. **3**

11. Use induction to prove that $\sum_{r=1}^{n} r^3 = \left[\frac{n}{2}(n+1)\right]^2$ for all integers $n \geq 1$. **4**

12. (*a*) The first term of an arithmetic sequence is 54, the last term is 19 and the sum of the terms of the sequence is 219.

Find the number of terms and the common difference. **3**

(*b*) The sum of the first and second terms of a geometric sequence is 1500, and the sum of the third and fourth terms is 375.

Find the two possible values of the common ratio and the corresponding values of the first term. **4**

Find the sum to five terms of both of the series. **2**

13. (*a*) Express $\dfrac{1}{4-y^2}$ in the form $\dfrac{A}{2-y}+\dfrac{B}{2+y}$, where *A* and *B* are constants. **2**

(*b*) Find $\displaystyle\int \sec^2 x\, dx$. **1**

(*c*) **Hence** find the particular solution of the equation $\dfrac{dy}{dx}=\dfrac{8-2y^2}{\cos^2 x}$,

given that $y(0)=\frac{2}{3}$. **7**

14. Show that $z=1+2i$ is a root of the equation $z^4-2z^3+9z^2-8z+20=0$. **2**

Write down a second root of the equation. **1**

Find the other two roots of the equation. **3**

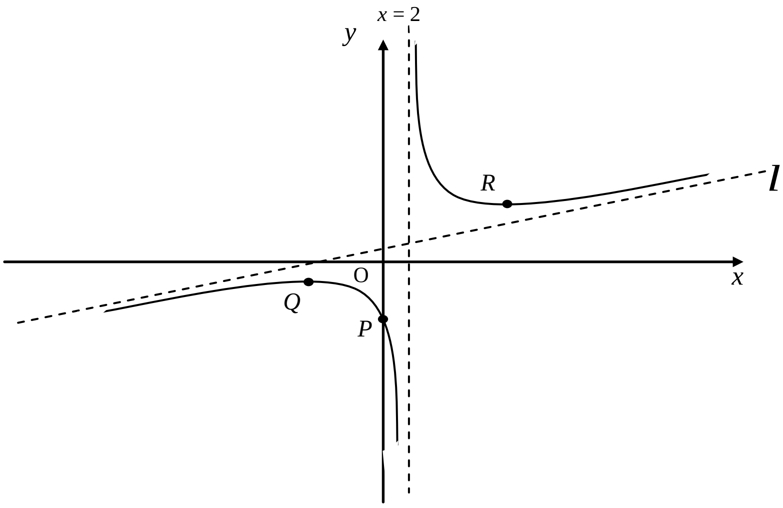
15. Given that $y=6^x\sqrt{1-2x}$, $\left(x\geq\frac{1}{2}\right)$, use logarithmic differentiation to find a formula for $\dfrac{dy}{dx}$ in terms of *x*. **4**

16. Use the substitution $x=\frac{2}{5}\sin\theta$ to show that

$$\int_0^{\frac{1}{5}} \frac{x}{\sqrt{4-25x^2}}\,dx = \frac{2-\sqrt{3}}{25}.$$ **6**

17. Part of the graph of the function $f(x) = \dfrac{x^2+12}{x-t}, x \neq t,$ is shown below.

The graph has a vertical asymptote with equation $x = 2$, a non-vertical asymptote l and crosses the y-axis at the point P. The points Q and R are the stationary points of the graph.



(a) Write down the value of t. **1**

(b) Determine algebraically the equation of the non-vertical asymptote, l. **3**

(c) Write down the coordinates of the point P where the graph of f crosses the y-axis. **1**

(d) Use calculus to find the coordinates and nature of the stationary points Q and R. **5**

(e) State the coordinates of the stationary points of the graph with equation

$$h(x) = 2|f(x)| - 1.$$ **2**

[*END OF QUESTION PAPER*]

Marking Scheme – Advanced Higher Grade 2014 / 2015
Prelim (Assessing Units 1 & 2)

	Give one mark for each •	Illustrations for awarding each mark
1 2005 Q1	(a) $f(x) = x^3 \tan 2x$ $f'(x) = 3x^2 \tan 2x + x^3\left(2\sec^2 2x\right)$ (b) $y = \dfrac{1 + x^2}{1 + x}$ $\dfrac{dy}{dx} = \dfrac{2x(1 + x) - (1 + x^2).1}{(1 + x)^2}$ $= \dfrac{x^2 + 2x - 1}{(1 + x)^2}$ *Alternative 1* $y = \dfrac{1 + x^2}{1 + x} = x - 1 + \dfrac{2}{1 + x}$ $\dfrac{dy}{dx} = 1 - \dfrac{2}{(1 + x)^2}$ or $1 - 2(1 + x)^{-2}$ *Alternative 2* $y = \dfrac{1 + x^2}{1 + x} = \left(1 + x^2\right)(1 + x)^{-1}$ $\dfrac{dy}{dx} = 2x(1 + x)^{-1} + \left(1 + x^2\right)(-1)(1 + x)^2$ $\dfrac{dy}{dx} = \dfrac{2x}{(1 + x)} - \dfrac{1 + x^2}{(1 + x)^2}$	 **1M, 1, 1** **1M,1** **1** **M1,1** **1** **M1,1** **1**
2 2006 Q6	$\dfrac{d}{dx}\left(x^4 - x^2 + 1\right) = 4x^3 - 2x$ Thus $\displaystyle\int \frac{12x^3 - 6x}{x^4 - x^2 + 1}\,dx = 3\int \frac{4x^3 - 2x}{x^4 - x^2 + 1}\,dx$ $= 3\ln\left(x^4 - x^2 + 1\right) + c$	**1** **1** **1**
3 2003 A9	Let $z = x + iy$. $z + i = (x + iy) + i = x + (1 + y)i$ $\therefore \quad \lvert z + i \rvert = \sqrt{x^2 + (1 + y)^2}$ $\therefore \quad x^2 + (1 + y)^2 = 4$ which is a circle, centre $(0, -1)$ radius 2. (The centre could be given as $-i$.)	 **1** **1** **1**

<table>
<tr><th></th><th>Give one mark for each •</th><th>Illustrations for awarding each mark</th></tr>
<tr><td>4</td><td>ans: $t = \pm\sqrt{2}$ 5 marks

• correct augmented matrix

• correct modified system

• correct upper triangular form

• correct equation
• correct answer</td><td>• $\begin{pmatrix} -2 & 3 & 1 & 7 \\ 1 & -2 & -2 & -3 \\ 2 & -2 & t^2 & 10 \end{pmatrix}$

• (E.g.) $\begin{pmatrix} 1 & -2 & -2 & -3 \\ 0 & -1 & -3 & 1 \\ 0 & 2 & t^2+4 & 16 \end{pmatrix}$

• $\begin{pmatrix} 1 & -2 & -2 & -3 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & t^2-2 & 18 \end{pmatrix}$

• $t^2 - 2 = 0$
• $t = \pm\sqrt{2}$ (both values required)</td></tr>
<tr><td>5

2010
Q10</td><td colspan="2">$f(x) = x^4 \sin 2x$ so
$f(-x) = (-x)^4 \sin(-2x)$ 1
$= -x^4 \sin 2x$ 1
$= -f(x)$
So $f(x) = x^4 \sin 2x$ is an odd function. 1

Note: a sketch given with a comment and correct answer, give full marks.
A sketch without a comment, gets a maximum of two marks.</td></tr>
<tr><td>6</td><td>ans: 12,288 5 marks

• correct general term

• simplifies correctly
• simplifies correctly
• correct value for r
• correct answer</td><td>• $\binom{6}{r}(2x^2y)^{6-r}\left(\frac{4}{xy^2}\right)^r$ or $\binom{6}{r}(2x^2y)^r\left(\frac{4}{xy^2}\right)^{6-r}$

(Ignore appearance of $\sum_{r=0}^{6}\ldots$)
• $\left(2^{6-r}\cdot 4^r \; or \; 2^{6+r}\right)$ or $\left(2^r\cdot 4^{6-r} \; or \; 2^{12-r}\right)$
• $x^{12-3r}y^{6-3r}$ or $x^{3r-6}y^{3r-12}$
• $r = 5$ or $r = 1$
• 12,288
(Ignore appearance of $(xy^3)^{-3}$ or $x^{-3}y^{-9}$)</td></tr>
</table>

	Give one mark for each •	Illustrations for awarding each mark
7 2008 Q7		

Working	Mark
$\int 8x^2 \sin 4x\,dx = 8x^2 \int \sin 4x\,dx - \int 16x\left(\int \sin 4x\,dx\right)dx$	**1M,1**
$= 8x^2\left(\frac{-1}{4}\cos 4x\right) - \int 16x \times \frac{-1}{4}\cos 4x\,dx$	**1**
$= -2x^2\cos 4x + 4\left[x\int \cos 4x\,dx - \int \frac{1}{4}\sin 4x\,dx\right]$	**1**
$= -2x^2\cos 4x + x\sin 4x + \frac{1}{4}\cos 4x + c$	**1**

8

Working	Mark	Illustration
$x = 2t + \frac{1}{2}t^2 \quad\Rightarrow\quad \frac{dx}{dt} = 2 + t$	**1**	
$y = \frac{1}{3}t^3 - 3t \quad\Rightarrow\quad \frac{dy}{dt} = t^2 - 3$	**1**	
$\frac{dy}{dx} = \frac{t^2 - 3}{2 + t}$	**1**	
$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2t(2+t) - (t^2 - 3)}{(2+t)^2} = \frac{t^2 + 4t + 3}{(2+t)^2}$	**1**	
$\frac{d^2y}{dx^2} = \frac{t^2 + 4t + 3}{(2+t)^2} \times \frac{1}{2+t} = \frac{t^2 + 4t + 3}{(2+t)^3}$	**1**	
Stationary points when $\frac{dy}{dx} = 0$, i.e.		
$t^2 - 3 = 0 \Rightarrow t = \pm\sqrt{3}$	**1**	
When $t = \sqrt{3}$, $\frac{d^2y}{dx^2} = \frac{3 + 4\sqrt{3} + 3}{(2 + \sqrt{3})^3} > 0$ which gives a minimum.	**1**	no marks for using a nature table
When $t = -\sqrt{3}$, $\frac{d^2y}{dx^2} = \frac{3 - 4\sqrt{3} + 3}{(2 - \sqrt{3})^3} < 0$ which gives a maximum.	**1**	no marks for using a nature table
At a point of inflexion, $\frac{d^2y}{dx^2} = 0$.	**1**	
In this case, that means		
$t^2 + 4t + 3 = (t + 1)(t + 3) = 0$		
and this has exactly two roots.	**1**	need to show 2 values exist
Note that this is a slimmed-down version of the complete story of points of inflexion.		

	Give one mark for each •	Illustrations for awarding each mark
9	**ans:** $\dfrac{dy}{dx}=\dfrac{4e^{4x}-ye^{xy}}{xe^{xy}+5e^{5y}}$ **4 marks** • starts correctly • continues correctly • differentiates correctly • simplifies correctly **ans:** $\frac{2}{3}e^4 \cong 36\cdot 4$ **2 marks** • substitutes correctly • correct answer	• $e^{xy}\left(\dfrac{d}{dx}(xy)\right)=\ldots$ • $\ldots=e^{4x}\left(\dfrac{d}{dx}(4x)\right)-e^{5y}\left(\dfrac{d}{dx}(5y)\right)$ • $ye^{xy}+xe^{xy}\dfrac{dy}{dx}=4e^{4x}-5e^{5y}\dfrac{dy}{dx}$ • $\dfrac{dy}{dx}=\dfrac{4e^{4x}-ye^{xy}}{xe^{xy}+5e^{5y}}$ • $\dfrac{4e^4-0\cdot e^0}{1\cdot e^0+5e^0}$ • $\frac{2}{3}e^4 \cong 36\cdot 4$
10	**ans:** $5+7i$ **3 marks** • knows how to deal with quotient • finds product correctly • correct answer	• $\dfrac{2-2i}{1+i}\cdot\dfrac{1-i}{1-i}$ • $5+5i$ • $5+7i$
11	**ans: Proof** **4 marks** • **starts correctly: proves result is true for n = 1** • **states correct assumption for n = k** • **continues proof correctly** • **completes proof correctly**	• $LHS=RHS=1 \Rightarrow$ True for n = 1 • Assume for n = k; i.e. that $\sum_{r=1}^{k} r^3=\left[\frac{k}{2}(k+1)\right]^2$ • $\sum_{r=1}^{k+1} r^3=\sum_{r=1}^{k} r^2+(k+1)^3$ $=\left[\frac{k}{2}(k+1)\right]^2+(k+1)^3=\left(\frac{k+1}{2}\right)^2\left(k^2+4(k+1)\right)$ $=\left(\frac{k+1}{2}\right)^2(k+2)^2=\left[\frac{k+1}{2}(k+2)\right]^2$ • Since result is true for n = 1 and, since (result is true for n = k) $\Rightarrow$ (result is true for n = k + 1), the result is true for all integers $n\ge 1$

	Give one mark for each •	Illustrations for awarding each mark
12 (*a*)	**ans:** $n = 6 \,\&\, d = -7$ **3 marks** • starts to substitute correctly into correct formula • correct number of terms • correct common difference	 • $219 = \frac{n}{2}\left(108 + (n-1)d\right)$ • $n = 6$ • $d = -7$
12 (*b*)	**ans:** $r = \frac{1}{2} \Rightarrow a = 1000 \,\&\, r = -\frac{1}{2} \Rightarrow a = 3000$ $r = \frac{1}{2} \Rightarrow S_5 = \frac{3875}{2} \,\&\, r = -\frac{1}{2} \Rightarrow S_5 = \frac{4125}{2}$ **6 marks** • starts process correctly • continues process correctly • correct values for *r* • correct values for *a* • finds sum correctly • finds sum correctly	 • $a + ar = 1500 \,\&\, ar^2 + ar^3 = 375$ • $\frac{1500}{a} = \frac{375}{ar^2}$ • $r^2 = \frac{375}{1000} \Rightarrow r = \frac{1}{2} \,\&\, r = -\frac{1}{2}$ • $a = 1000 \,\&\, a = 3000$ • $r = \frac{1}{2} \Rightarrow S_5 = \frac{3875}{2}$ • $r = -\frac{1}{2} \Rightarrow S_5 = \frac{4125}{2}$

	Give one mark for each •	Illustrations for awarding each mark
13 (*a*)	**ans:** $\frac{1}{4(2-y)}+\frac{1}{4(2+y)}$ **2 marks** • correct answer • correct answer	 • $A=\frac{1}{4}$ • $B=\frac{1}{4}$
13 (*b*)	**ans:** $\tan x+C$ **1 mark** • correct answer	 • $\tan x+C$ (accept $\tan x$)
13 (*c*)	**ans:** $y=\frac{4e^{\tan x}-2}{1+2e^{\tan x}}$ *or equivalent* **7 marks** • rearranges correctly • substitutes correctly • integrates correctly • rearranges & starts to solve correctly • correct general solution • substitutes & starts to evaluate correctly • solves correctly	 • $\frac{1}{2}\int\frac{dy}{4-y^2}=\int\sec^2 xdx$ • $\frac{1}{2}\int\left(\frac{1}{4(2-y)}+\frac{1}{4(2+y)}\right)dy=\int\sec^2 xdx$ • $\frac{1}{8}\left(-\ln\lvert 2-y\rvert+\ln\lvert 2+y\rvert\right)=\tan x+C$ • $e^{\ln\left\lvert\frac{2+y}{2-y}\right\rvert}=e^{8\tan x+c}$ • $y=\frac{2Ae^{8\tan x}-2}{1+Ae^{8\tan x}}$ • $\frac{2}{3}=\frac{2Ae^{0}-2}{1+Ae^{0}}$ • $A=2$

	Give one mark for each •	Illustrations for awarding each mark
14	**ans:** Proof **2 marks** • starts to show that $1+2i$ is a root correctly • completes proof **ans:** $1-2i$ **1 mark** • correct answer **ans:** $\pm 2i$ **3 marks** • starts correctly • continues correctly • finds correct roots	• $1+2i\overline{\left)\begin{array}{ccccc} 1 & -2 & 9 & -8 & 20 \\ & 1+2i & -5 & 4+8i & -20 \end{array}\right.}$ (or equivalent) • $\ldots = 0$ • $1-2i$ • $\begin{array}{cccc} 1 & 0 & 4 & 0 \end{array}$ $1-2i\overline{\left)\begin{array}{cccc} 1 & -1+2i & 4 & -4+8i \\ & 1-2i & 0 & 4-8i \end{array}\right.}$ (or equivalent) • $z^2+4=0$ • $z=\pm 2i$
15	**ans:** $\frac{dy}{dx}=6^x\sqrt{1-2x}\left(\ln 6-\frac{1}{1-2x}\right)$ **4 marks** • starts correctly • manipulates correctly • differentiates correctly • correct answer	• $\ln y=\ln\left(6^x\sqrt{1-2x}\right)$ • $\ln y = x\ln 6+\frac{1}{2}\ln(1-2x)$ • $\frac{1}{y}\frac{dy}{dx}=\ln 6+\frac{1}{2}\cdot\frac{1}{1-2x}\cdot(-2)$ • $\frac{dy}{dx}=6^x\sqrt{1-2x}\left(\ln 6-\frac{1}{1-2x}\right)$
16	**ans: Proof** **6 marks** • **substitutes correctly** • **changes limits correctly** • **simplifies correctly** • **integrates correctly** • **substitutes correctly** • **simplifies correctly**	• $\int\frac{\frac{2}{5}\sin\theta\cdot\frac{2}{5}\cos\theta}{\sqrt{4-25\left(\frac{2}{5}\sin\theta\right)^2}}\,d\theta$ • $\int_0^{\frac{\pi}{6}}\ldots$ • $\frac{4}{25}\cdot\frac{1}{2}\int\frac{\sin\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta=\frac{2}{25}\int\sin\theta d\theta$ • $\frac{2}{25}[-\cos\theta]$ • $\frac{2}{25}\left(-\cos\frac{\pi}{6}+\cos 0\right)$ • $\frac{2}{25}\left(-\frac{\sqrt{3}}{2}+1\right)=\frac{2-\sqrt{3}}{25}$

	Give one mark for each •	Illustrations for awarding each mark
17(a)	**ans:** $t = 2$ **1 mark** • correct value	• $t = 2$
17(b)	**ans:** $y = x + 2$ **3 marks** • correct method • continues correctly • correct answer	• $x - 2\overline{)x^2 + 12}$ • $y = x + 2 + \dfrac{16}{x-2}$ • $y = x + 2$
17(c)	**ans:** $P(0,-6)$ **1 mark** • correct coordinates	• $(0,-6)$
17(d)	**ans:** $Q(-2,-4) \rightarrow Maximum\,T.P.$ & $R(6,12) \rightarrow Minimum\,T.P.$ **5 marks** • differentiates correctly • solves $f'(x) = 0$ correctly • correct y-coordinates • correct formula for second derivative • correct nature of both points	• $f'(x) = 1 - \dfrac{16}{(x-2)^2}$ • $x = -2, 6$ • $Q(-2,-4)$ & $R(6,12)$ • $f''(x) = \dfrac{32}{(x-2)^3}$ • $f''(-2) < 0 \Rightarrow Maximum\,T.P.$ & $f''(6) > 0 \Rightarrow Minimum\,T.P.$
17(e)	**ans:** $(-2,7), (6,23)$ **2 marks** • correct point • correct point	• $(-2,7)$ • $(6,23)$

<u>TOTAL MARKS = 100</u>