Proof Theory

Skill	Achieved ?
Know that a ***sentence*** is any concatenation of letters or symbols that has a meaning	
Know that something is ***true*** if it appears psychologically convincing according to current knowledge	
Know that something is ***false*** if it is not true	
Know that truth in real-life is often time-dependent; for example, ' The president of America is Ronald Reagan ' was true but is presently false	
Know that truth in mathematics is not time-dependent	
Know that not every sentence is true or false, for example: Who is that person ? Walk ! This sentence is false	
Know that a ***statement*** (aka ***proposition***) is a sentence that is either true or false, for example: All fish are orange in colour (F) The Milky Way is a galaxy (T) 4 is a prime number (F) 26 is divisible by 13 (T)	
Know that a ***compound statement*** is one obtained by combining 2 or more statements, especially by using 'and' or 'or', for example: ' The Milky Way is a galaxy' and '4 is a prime number ' (F) ' The Milky Way is a galaxy' or '4 is a prime number ' (T)	
Know that the ***negation*** of a statement S is the statement ' not S ' ($\sim S$), and is such that if S is true, then the negation is false (or, if S is false, then the negation is true)	
Know that a ***universal statement*** is one that refers to all elements of a set	
Know that an ***existential statement*** is one that refers to the existence of at least one element of a set	

Know that a ***proof*** is a logically convincing argument that a given statement is true	
Know that an ***axiom*** (aka ***assumption*** or ***hypothesis*** or ***postulate*** or ***premise***) is a statement that is taken to be true (not requiring proof) and used before the end of an argument	
Know that a ***conclusion*** (aka ***thesis***) is a statement that is reached at the end of an argument	
Know that the statement, ' If A, then B ' is called a ***(material) conditional*** (aka ***if, then statement*** or ***conditional*** or ***implication***) and written $A \Rightarrow B$ (read, ' A ***implies*** B '); A is called the ***implicant*** (aka ***antecedent***) and B the ***implicand*** (aka ***consequent***)	
Know that a conditional is true except when A is true and B is false (a true statement cannot imply a false one)	
Know that A and B are ***equivalent statements*** if $A \Rightarrow B$ and $B \Rightarrow A$, i.e. if $A \Leftrightarrow B$ (read, ' A ***if and only if*** B '); the statement $A \Leftrightarrow B$ is called a ***biconditional*** or ***double implication***	
Know that the ***converse*** of the statement $A \Rightarrow B$ is $B \Rightarrow A$	
Know that the ***inverse*** of the statement $A \Rightarrow B$ is $\sim A \Rightarrow \sim B$	
Know that the ***contrapositive*** of the statement $A \Rightarrow B$ is $\sim B \Rightarrow \sim A$ and is equivalent to the statement $A \Rightarrow B$	
Know that an ***example*** (aka ***instance***) is something that satisfies a given statement	
Know that if an existential statement is true, then it can be proved by citing an example	
Prove existential statements by citing an example, such as: $\exists n \in \mathbb{N}$ such that $n^2 + 1$ is even	
Know that a ***counterexample*** is an exception to a proposed statement	
Know that to ***disprove*** a statement means proving a statement false	
Know that if a universal statement is false, then it can be disproved by citing a counterexample	
Disprove a universal statement by finding a counterexample, such as: $n^2 + n$ is a multiple of 3 $(\forall n \in \mathbb{N})$ $n^3 + n + 5$ is prime $(\forall n \in \mathbb{N})$ m^2 divisible by 4 $\Rightarrow$ m divisible by 4 $(\forall m \in \mathbb{N})$	

$\sqrt{a} + \sqrt{b}$ irrational $\Rightarrow$ $\sqrt{ab}$ irrational ($\forall\, a, b \in \mathbb{N}$) k prime $\Rightarrow$ $2^k - 1$ prime ($\forall\, k \in \mathbb{N}$)	
Know that a ***direct proof*** is one where a statement S is proved by starting with a statement and assumptions and proceeding through a chain of logical steps to reach the conclusion S	
Use direct proof to prove statements about odd and even numbers, for example: The square of an even number is even The square of an odd number is odd The product of an even and an odd number is odd The cube of an odd number is odd The cube of an odd number plus the square of an even number is odd	
Use direct proof to prove statements that involve a finite number of cases, for example: $n \in \mathbb{N} \Rightarrow n^2 + n$ is even n odd $\Rightarrow$ $n^2 - 1$ is divisible by 8	
Use direct proof to prove other statements, for example: $n^2 + n + 3$ is always odd	
Know that an ***indirect proof*** is a proof that involves negating a statement and are of 2 types	
Know that ***proof by contrapositive*** is an indirect proof technique of proving a conditional by assuming the negation of the consequent and proving the negation of the antecedent	
Prove statements by contrapositive, for example: n odd $\Rightarrow$ n^2 odd	
Know that ***proof by contradiction*** is a indirect proof technique of proving a conditional by assuming the antecedent and the negation of the consequent and reaching a contradiction by negating the antecedent	
Prove statements by contradiction, for example: $\sqrt{2}$ is irrational	

$\sqrt{5}$ is irrational $\frac{1}{3}\left(4\sqrt{3} - 1\right)$ is irrational x is irrational $\Rightarrow$ $2 + x$ is irrational $a, b \in \mathbb{R}$ and $a + b$ is irrational $\Rightarrow$ a or b is irrational $a, b \in \mathbb{N}$ and $\sqrt{ab}$ is irrational $\Rightarrow$ $\sqrt{a} + \sqrt{b}$ is irrational	
Know that the technique of ***proof by mathematical induction*** (aka ***induction***) involves proving a statement, denoted P(n) regarding the set of all natural numbers (or all natural numbers except a finite subset thereof)	
Know that proof by induction involves verifying (i) the ***Base case*** (usually $n = 1$) (ii) proving the ***Inductive step*** (using the ***inductive hypothesis***) by verifying that $P(n) \Rightarrow P(n + 1)$	
Prove statements by induction where P(n) is, for example: $4^n - 1$ is divisible by 3 $(\forall n \in \mathbb{N})$ $2^{3n} - 1$ is divisible by 7 $(\forall n \in \mathbb{N})$ p is odd $\Rightarrow$ p^n is odd $(\forall n \in \mathbb{N})$ $(1 + a)^n \geq 1 + na$ $(\forall n \in \mathbb{N})$ $2^n > n$ $(\forall n \in \mathbb{N})$ $3^n > 2^n$ $(\forall n \in \mathbb{N})$ $2^n > n^2$ $(\forall n \in \mathbb{N} \setminus \{1, 2, 3, 4\})$ $n! > 2^n$ $(\forall n \in \mathbb{N} \setminus \{1, 2, 3\})$	

Further Differentiation

Skill	Achieved ?
Differentiate the inverse of a function f using the formula: $$D(f^{-1}) = \frac{1}{(Df)\circ f^{-1}}$$	
Know that when $y = f(x)$, the above formula is written in Leibniz notation as: $$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$	
Differentiate inverse functions using the above formula	
Know that: $$D(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$$ $$D(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$ $$D(\tan^{-1} x) = \frac{1}{1 + x^2}$$	
Differentiate functions such as: $$f(x) = (2 + x)\tan^{-1}\sqrt{x - 1}$$ $$w(x) = 2\tan^{-1}\sqrt{1 + x}$$ $$a(x) = \cos^{-1}(3x)$$ $$g(x) = \frac{\tan^{-1} 2x}{1 + 4x^2}$$	
Know the meanings of ***implicit equation*** and ***implicit function***	
Know the meaning of ***implicit differentiation***	
Determine whether or not a given point lies on a curve defined by an implicit equation	
Given the x-coordinate of a point on a curve defined by an implicit equation, determine the y-coordinate	
Given the y-coordinate of a point on a curve defined by an	

AH Checklist (Unit 2)

implicit equation, determine the x-coordinate	
Find $\frac{dy}{dx}$ for implicit equations such as: $xy + y^2 = 2$ $2y^2 - 2xy - 4y + x^2 = 2$ $xy - x = 4$ $x = \cot y$ $xy^2 + 3x^2y = 4$ $\frac{x^2}{y} + x = y - 5$ $\ln y = \sin x - \cos y$ $e^{2x+y} = \ln(7y - x)$ $x\tan y = e^{3x}$ $\sin^{-1}x + \cos^{-1}y = 2x$ $y + e^y = x^2$	
Work out the second derivative of an implicitly defined function	
Use implicit differentiation to find $\frac{d^2y}{dx^2}$ for implicit equations such as: $xy - x = 4$	
Use implicit differentiation to find the equation of the tangent to a curve written in implicit form, such as: $y^3 + 3xy = 3x^2 - 5$ at $(2, 1)$ $xy^2 + 3x^2y = 4$ at $x = 1$	
Know that ***logarithmic differentiation*** involves taking the (usually, natural) logarithm of a function and then differentiating	
Use logarithmic differentiation to differentiate functions of the form:	

$y = (x+1)^2 (x+2)^{-4}$ $y = \dfrac{(3x+1)^{2/3}(2x-5)^{3/2}}{(4x+7)^{1/4}}$ $y = x e^{-2x} \sin x$ $y = \dfrac{e^x \cos x}{x}$ $y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$	
Use logarithmic differentiation to differentiate functions of the form: $y = 3^x$ $y = x^x$ $y = (\sin x)^x$ $y = \pi^{x^2}$ $y = 5^{e^x}$ $y = e^{\cos^2 x}$ $y = 4^{x^2+1}$ $y = x^{\sin x}$ $y = (x+3)^{x-2}$ $y = x^{2x^2+1}$	
Know that a curve $y = f(x)$ can be defined parametrically by 2 functions, $x(t)$ and $y(t)$, called ***parametric functions*** (aka ***parametric equations***) ***with parameter t***	
Determine whether or not a point lies on a parametrically defined curve	
Given a pair of parametric equations, know that:	

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$	
Know that the above formula is sometimes written: $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$	
Calculate $\frac{dy}{dx}$ for parametric functions such as: $x = 2\sec\theta,\ y = 3\sin\theta$	
Find the gradient or equation of a tangent line to a parametrically defined curve given a point on the curve, such as: $x = t^2 + t - 1,\ y = 2t^2 - t + 2$ at $(-1, 5)$	
Find the gradient or equation of a tangent line to a parametrically defined curve given a value for the parameter, such as: $x = 5\cos\theta,\ y = 5\sin\theta$ when $\theta = \frac{\pi}{4}$	
Given a pair of parametric equations, know the 2 formulae for calculating the second derivative: $\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \times \frac{1}{\dot{x}}$	
Work out $\frac{d^2y}{dx^2}$ for parametric functions such as: $x = \cos 2t,\ y = \sin 2t$	
Investigate points of inflexion for parametric functions such as: $x = 2t - 3 - \frac{1}{t},\ y = t - 1 - \frac{2}{t},$ $y = t^3 - \frac{5}{2}t^2,\ x = \sqrt{t}$	

Applications of Differentiation

Skill	Achieved ?
Know that ***planar motion*** means motion in 2 dimensions, described in Cartesian coordinates by 2 functions of time $x(t)$ and $y(t)$ (the dependence on t usually being suppressed)	
Know that the displacement of a particle at time t in a plane is described by the ***displacement vector*** : $\mathbf{s}(t) \overset{def}{=} (x(t), y(t)) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$	
Calculate the ***magnitude of displacement***, aka ***distance (from the origin)***, using: $\lvert \mathbf{s}(t) \rvert \overset{def}{=} \sqrt{x^2 + y^2}$	
Know that the velocity of a particle at time t in a plane is described by the ***velocity vector*** : $\mathbf{v}(t) \overset{def}{=} \frac{d\mathbf{s}}{dt} = (\dot{x}(t), \dot{y}(t)) = \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j}$	
Calculate the velocity vector given the displacement vector	
Calculate the ***magnitude of velocity***, aka ***speed***, at any instant of time t using: $\lvert \mathbf{v}(t) \rvert \overset{def}{=} \sqrt{\dot{x}^2 + \dot{y}^2}$	
Calculate the ***direction of motion*** (aka ***direction of velocity***), θ, at any instant of time t using: $\tan\theta = \frac{\dot{y}}{\dot{x}}$ where θ is the angle between $\dot{x}\,\mathbf{i}$ and $\mathbf{v}$	
Know that the acceleration of a particle at time t in a plane is described by the ***acceleration vector*** : $\mathbf{a}(t) \overset{def}{=} \frac{d\mathbf{v}}{dt} = (\ddot{x}(t), \ddot{y}(t)) = \ddot{x}(t)\,\mathbf{i} + \ddot{y}(t)\,\mathbf{j}$	
Calculate the acceleration vector given the velocity vector or displacement vector	

Calculate the ***magnitude of acceleration*** using: $\lvert \mathbf{a}(t) \rvert \overset{def}{=} \sqrt{\ddot{x}^2 + \ddot{y}^2}$	
Calculate the ***direction of acceleration***, η, at any instant of time t using: $\tan \eta = \frac{\ddot{y}}{\ddot{x}}$ where η is the angle between $\ddot{x}\,\mathbf{i}$ and $\mathbf{a}$	
Know that ***related rates of change*** refers to when y is a function of x and both x and y are each functions of a third variable u	
Know that related rates of change are linked via the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{dy}{du} \div \frac{dx}{du}$	
Know that related rates of change problems may involve use of: $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$	
Solve problems involving related rates of change, for example, if a spherical balloon is inflated at a constant rate of 240 cubic centimetres per second, find (i) the rate at which the radius is increasing when the radius is 8 cm (ii) the rate at which the radius is increasing after 5 seconds	
Know that in related rates of change problems, the relationship between x and y may be an implicit one	

Further Integration

Skill	Achieved ?
Know the standard integrals: $$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$ $$\int \frac{1}{a^2 + x^2}\,dx = \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$	
Know the special cases of the above 2 standard integrals: $$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$$ $$\int \frac{dx}{1 + x^2} = \tan^{-1} x + C$$	
Know that rational functions can be integrated using the technique of ***integration by partial fractions***	
Use integration by partial fractions to find or evaluate integrals such as: $$\int \frac{x^3}{x^2 - 1}\,dx$$ $$\int_0^1 \frac{1}{x^2 - x - 6}\,dx$$ $$\int_0^k \frac{1}{x^3 + x}\,dx$$ $$\int \frac{12x^3 - 6x}{x^4 - x^2 + 1}\,dx$$ $$\int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)}\,dx$$	

$$\int_1^2 \frac{12x^2 + 20}{x(x^2 + 5)} \, dx$$ $$\int_1^2 \frac{3x + 5}{(x + 1)(x + 2)(x + 3)} \, dx$$	
Know that ***integration by parts*** is the technique that is used to integrate some products of functions	
Know the ***integration by parts formula*** : $$\int u\,(D\,v) = u\,v - \int (D\,u)\,v$$	
Know that integration by parts involves differentiating one function, u, and integrating the other, Dv	
Use integration by parts to find or evaluate integrals such as: $$\int_0^{\pi/4} 2x \sin 4x \, dx$$ $$\int_0^1 \ln(1 + x) \, dx$$ $$\int_0^1 x\,e^{-x} \, dx$$ $$\int_0^{1/2} \sin^{-1} x \, dx$$ $$\int_0^{\pi/4} \cos^4 x \, dx$$ $$\int_0^{1/2} \tan^{-1} 2x \, dx$$	

$\int x^2 \ln x \quad dx$ $\int_0^1 x \tan^{-1} x^2 \quad dx$	
Use a double application of integration by parts to find integrals, for example: $\int x^2 \sin x \quad dx$ $\int 8x^2 \sin 4x \quad dx$ $\int 3x^2 \cos 2x \quad dx$	
Use integration by parts to find integrals by a 'cyclical procedure', for example: $\int e^x \sin x \quad dx$ $\int e^x \cos x \quad dx$	
Know the meaning of ***reduction formula***	
Obtain reduction formulae for integrals, for example: $\int x^n e^{ax} \ dx \equiv I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1} \qquad (a \in \mathbb{R} \setminus \{0\})$ $\int_0^1 x^n e^{-x} \ dx \equiv I_n = -\frac{1}{e} + n I_{n-1}$ $\int \sin^n x \ dx \equiv I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$ $\int \cos^n x \ dx \equiv I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$	
Obtain the general solution of simple differential equations such as:	

$\frac{dy}{dx} = \frac{1}{25 + x^2}$ $\frac{dy}{dx} = \sin nx \quad (n \in \mathbb{N})$ $\frac{dy}{dx} = \frac{7}{x^2 - 4}$ $\frac{dy}{dx} = \cot 5x$	
Obtain particular solutions of simple differential equations, given initial conditions	
Solve simple differential equations in practical contexts	
Know that a ***separable differential equation*** is one that can be written in the form: $\frac{dy}{dx} = f(x)\,g(y)$	
Obtain the general solution of separable DEs, for example: $\frac{dM}{dt} = kM$ $\frac{dy}{dx} = \frac{y}{x}$ $\frac{dV}{dt} = V(10 - V)$ $y\frac{dy}{dx} - 3x = x^4$ $\frac{dG}{dt} = \frac{25k - G}{25} \quad (k \text{ constant})$ $x^2 e^y \frac{dy}{dx} = 1$ $\frac{dy}{dx} = 3(1 + y)\sqrt{1 + x}$	
Given initial conditions, obtain a particular solution of a separable DE	

Complex Numbers

Skill	Achieved ?
Know that a ***complex number*** is a number of the form (aka ***Cartesian form***), $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$; x is called the ***real part of z (Re(z))*** and y the ***imaginary part of z (Im(z))***	
Know that the set of all complex numbers is defined as: $\mathbb{C} \overset{def}{=} \{ x + iy : x, y \in \mathbb{R}, i^2 = -1 \}$	
Know that complex numbers are equal if their real parts are equal and their imaginary parts are equal, and conversely	
Know that: $\sqrt{-a} = i\sqrt{a}$	
Solve any quadratic equation, for example: $z^2 - 2z + 5 = 0$	
Add or subtract complex numbers by adding or subtracting the corresponding real parts and the corresponding imaginary parts: $(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$	
Multiply complex numbers according to the rule: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$	
Know that the ***complex conjugate*** of $z = x + iy$ is defined as: $\overline{z} \overset{def}{=} x - iy$	
Know that the complex conjugate satisfies: $z\overline{z} = x^2 + y^2$	
Use the complex conjugate to divide any 2 complex numbers	
Calculate the square root of any complex number	
Know that a complex number z can be represented as a point P (or coordinate or vector) in the ***complex plane*** (aka ***Argand plane***); with P plotted, the result is called an ***Argand diagram*** (aka ***Wessel diagram***)	
Know that the horizontal axis in the complex plane is called the	

real axis, while the vertical axis is called the ***imaginary axis***	
Plot a complex number written in Cartesian form in the Argand plane	
Plot the complex conjugate of a given complex number in Cartesian form in the Argand plane	
Know that the ***modulus of z*** is the distance from the origin to P and defined by: $r \equiv \lvert z \rvert \overset{def}{=} \sqrt{x^2 + y^2}$	
Calculate the modulus of any complex number	
Know that the angle in the interval $(-\pi, \pi]$ from the positive x – axis to the ray joining the origin to P is called the ***(principal) argument of z*** and defined by: $\theta \equiv \arg z \overset{def}{=} \tan^{-1}\left(\frac{y}{x}\right)$	
Know that a complex number has infinitely many arguments, but only 1 principal argument, the 2 types of argument being related by: $\operatorname{Arg} z \overset{def}{=} \{\arg z + 2\pi n : n \in \mathbb{Z}\}$	
Calculate an argument and the principal argument of any complex number	
Know that, from an Argand diagram: $x = r\cos\theta \ , \ y = r\sin\theta$	
Know that a complex number can be written in ***polar form***: $z = r(\cos\theta + i\sin\theta) \equiv r\operatorname{cis}\theta$	
Plot a complex number written in polar form in the complex plane	
Given a complex number in Cartesian form, write it in polar form	
Given a complex number in polar form, write it in Cartesian form	
Identify, describe and sketch loci in the complex plane, for example: $\lvert z \rvert = 6$ $\lvert z \rvert \leq 4$ $\lvert z - 2 \rvert = 3$ $\lvert z + i \rvert = 2$	

$\lvert z - 2 + 4i \rvert = 1$ $\lvert z - 2 \rvert = \lvert z + i \rvert$ $\arg z = \frac{2\pi}{3}$	
Know that when multiplying 2 complex numbers z and w, the following results hold: $\lvert zw \rvert = \lvert z \rvert \lvert w \rvert$, $\text{Arg } zw = \text{Arg } z + \text{Arg } w$	
Know that when dividing 2 complex numbers z and w, the following results hold: $\left\lvert \frac{z}{w} \right\rvert = \frac{\lvert z \rvert}{\lvert w \rvert}$, $\text{Arg } \frac{z}{w} = \text{Arg } z - \text{Arg } w$	
Multiply and divide 2 or more complex numbers in polar form using the above rules for the modulus and argument	
Know ***de Moivre's Theorem*** (for $k \in \mathbb{R}$): $z = r(\cos\theta + i\sin\theta) \Rightarrow z^k = r^k(\cos k\theta + i\sin k\theta)$	
Use de Moivre's Theorem to evaluate powers of a complex number written in polar form, writing the answer in Cartesian form	
By considering the binomial expansion of $(\cos\theta + i\sin\theta)^n$ ($n \in \mathbb{N}$), use de Moivre's Theorem to obtain expressions for $\cos n\theta$ and $\sin n\theta$, in particular, $\cos 3\theta$, $\sin 3\theta$, $\cos 4\theta$ and $\sin 4\theta$	
Know that if $w = r(\cos\theta + i\sin\theta)$, then the n solutions of the equation $z^n = w$ are given by: $z_k = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$ $(k = 0, 1, 2, \ldots, n-1)$	
Know that plotting the solutions $z^n = w$ with w given as above on an Argand diagram illustrates that the n roots are equally spaced on a circle centre (0, 0), radius $r^{1/n}$, with the angle between any 2 consecutive solutions being $\frac{2\pi}{n}$	
Find (and plot) the roots of any complex number using the above formula	

Know that solving the equation $z^n = 1$ gives the ***roots of unity***	
Know that the n roots of unity ($n > 1$) satisfy the equation: $$\sum_{k=0}^{n-1} z_k = 0$$	
Verify that a given complex number is a root of a cubic or quartic	
Know that a repeated root (occurring m times) of a polynomial p is called a root of ***multiplicity m***	
Know that the ***Fundamental Theorem of Algebra*** states that every (non-constant) polynomial with complex coefficients has at least one complex root	
Know that the Fundamental Theorem of Algebra implies that every polynomial of degree n (≥ 1) with complex coefficients has exactly n complex roots (including multiplicities)	
Know that a polynomial p of degree at least 1 with complex coefficients can be factorised into a product of n linear factors: $$p(z) = \prod_{r=1}^{n} (z - z_r)$$	
Know that if a polynomial p of degree n with all coefficients real has a non-real root, then the conjugate of this root is also a root of p	
Know that a polynomial of degree n with all coefficients real can be factorised into a product of real linear factors and real irreducible quadratic factors: $$p(z) = \prod_{r=1}^{t} (z - d_r) \times \prod_{s=1}^{(n-t)/2} (a_s z^2 + b_s z + c_s)$$	
Solve cubic and quartic equations which have all coefficients real, for example: $$z^4 + 4z^3 + 3z^2 + 4z + 2 = 0$$ $$z^3 + 3z^2 - 5z + 25 = 0$$ $$z^3 - 18z + 108 = 0$$	
Factorise a cubic or quartic into a product of linear factors	
Factorise a polynomial with all coefficients real into a product of real linear factors and real irreducible quadratic factors	

Sequences and Series

Skill	Achieved ?
Know that a ***series*** (aka ***infinite series***) is the terms of a sequence added together	
Know that the ***sum to n terms*** (aka ***sum of the first n terms*** aka ***n th partial sum***) ***of a sequence*** is: $S_n \stackrel{def}{=} \sum_{r=1}^{n} u_r$	
Calculate the sum to n terms of a given sequence	
Given a formula for S_n, calculate u_1, u_2 etc. using the prescription: $u_n = S_{n+1} - S_n$	
Know that the ***sum to infinity*** (aka ***infinite sum***) of a sequence is the limit (if it exists) as $n \to \infty$ of the n^{th} partial sums, i.e. : $S_\infty \stackrel{def}{=} \lim_{n\to\infty} S_n$	
Know that an infinite series ***converges*** (aka ***is summable***) if S_∞ exists; otherwise, the series ***diverges***	
Know that not every sequence has a sum to infinity	
Know that the symbol a is traditionally used to denote the first term of a sequence	
Know that an ***arithmetic sequence*** is one in which the difference of any 2 successive terms is the same, this latter being called the ***common difference (d)***	
Show that a given sequence of numbers or expressions forms an arithmetic sequence	
Know that the n^{th} term of an arithmetic sequence is given by: $u_n = a + (n - 1)\, d \qquad (a \in \mathbb{R}\, ,\, d \in \mathbb{R} \setminus \{0\})$	
Given a, n and d for an arithmetic sequence, calculate u_n	
Given a, n and u_n for an arithmetic sequence, calculate d	
Given a, u_n and d for an arithmetic sequence, calculate n	
Given u_n, n and d for an arithmetic sequence, calculate a	
Given 2 specific terms of an arithmetic sequence, find the first term and common difference	
Know that the sum to n terms of an arithmetic series is given by:	

$S_n = \frac{n}{2}\left(2a + (n - 1)d\right)$	
Given a, n and d for an arithmetic sequence, calculate S_n	
Given a, n and S_n for an arithmetic sequence, calculate d	
Given a, S_n and d for an arithmetic sequence, calculate n	
Given S_n, n and d for an arithmetic sequence, calculate a	
Obtain sums of arithmetic series, such as: $8 + 11 + 14 + \ldots + 56$	
Know that the sum to n terms of an arithmetic sequence can always be written in the form: $S_n = P n^2 + Q n \quad (P \in \mathbb{R} \setminus \{0\}, Q \in \mathbb{R})$	
Given a formula for S_n for an arithmetic series, calculate u_1, u_2 etc.	
Know that no arithmetic series has a sum to infinity	
Solve contextual problems involving arithmetic sequences and series	
Know that a ***geometric sequence*** is one in which the ratio of any 2 successive terms is the same, this latter being called the ***common ratio (r)***	
Know that the n^{th} term of a geometric sequence is given by: $u_n = a r^{n-1} \quad (a \in \mathbb{R} \setminus \{0\}, r \in \mathbb{R} \setminus \{0, 1\})$	
Show that a given sequence of numbers or expressions forms an arithmetic sequence	
Given a, n and r for a geometric sequence, calculate u_n	
Given a, n and u_n for a geometric sequence, calculate r	
Given a, u_n and r for a geometric sequence, calculate n	
Given u_n, n and r for a geometric sequence, calculate a	
Given 2 specific terms of a geometric sequence, find the first term and common ratio	
Know that the sum to n terms of a geometric series is given by: $S_n = \frac{a(1 - r^n)}{1 - r}$	
Given a, n and r for a geometric sequence, calculate S_n	
Given S_n, n and r for a geometric sequence, calculate a	
Given a, S_n and r for a geometric sequence, calculate n	
Given a, n and S_n for a geometric sequence, calculate r	
Obtain sums of geometric series, such as:	

$50 - 20 + 8 - \dots$ (to 8 terms)	
Know that a geometric series may or may not have a sum to infinity	
Know that S_∞ exists for a geometric series if $\lvert r \rvert < 1$	
Know that the sum to infinity of a geometric series is given by: $$S_\infty = \frac{a}{1 - r}$$	
Given a and r for a geometric sequence, calculate S_∞	
Given S_∞ and r for a geometric sequence, calculate a	
Given S_∞ and a for a geometric sequence, calculate r	
Express a recurring decimal as a geometric series and as a fraction	
Know that a ***power series*** is an expression of the form: $$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \quad (a_i \in \mathbb{R})$$	
Know that if $\lvert x \rvert < 1$, then $$(1 - x)^{-1} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots \overset{def}{=} \sum_{i=0}^{\infty} x^i$$	
Expand as a power series other reciprocals of binomial expressions, stating the range of values for which the expansion is valid, for example: $$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \dots = \sum_{i=0}^{\infty} (-1)^i x^i$$ $$\frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + \dots = \sum_{i=0}^{\infty} x^{2i}$$ $$\frac{1}{1 + 3x} = 1 - 3x + 9x^2 - 27x^3 + \dots = \sum_{i=0}^{\infty} (-1)^i 3^i x^i$$	
Expand as a geometric series (stating the range of validity of the expansion), more complicated reciprocals of binomials, for example: $$(3 + 4x)^{-1}$$ $$(\sin x - \cos x)^{-1}$$	

$(\cos 2x)^{-1}$	
Know that: $e \overset{def}{=} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots = \sum_{b=0}^{\infty}\frac{1}{b!}$	
Know that: $e^x \overset{def}{=} \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$	
Find other limits using the above limit, for example: $\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n = e^5$	
Know that if a power series tends to a limit, then the power series can be differentiated and the differentiated series tends to the derivative of the limit	
Differentiate a power series and find a formula for the limit, for example: $\frac{d}{dx}(1 - x + x^2 - x^3 + \ldots) = -\left(\frac{1}{1+x}\right)^2$	
Know that if a power series tends to a limit, then the power series can be integrated and the integrated series tends to the integral of the limit	
Integrate a power series and find a formula for the limit, for example: $\int (1 - x + x^2 - x^3 + \ldots)\,dx = \ln(1 + x)$	
Solve contextual problems involving arithmetic sequences and series	
Know the results: $\sum_{r=1}^{n} 1 = n$ $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$	
Explore other results such as:	

AH Checklist (Unit 2)

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$	
Evaluate finite sums using a combination of the above 2 finite sums, for instance: $\sum_{k=1}^{n} (11 - 2k)$ $\sum_{r=1}^{n} (4 - 6r)$	
Evaluate finite sums that don't start at 1, for example: $\sum_{k=3}^{n} (7 - k)$ $\sum_{r=5}^{17} (3r + 11)$	